PROBLEMS IN THE REPRESENTATION THEORY OF ALGEBRAIC GROUPS

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General Remarks

This thesis is in two parts. Part A is concerned with a problem in the representation theory of semisimple algebraic groups. In part B we are interested in the endomorphism rings of comodules, in particular in the endomorphism ring of a certain induced comodule. The parts are more or less self contained each having its own introduction and references. There is however an underlying theme of induction and so the exposition of 2, (A) is relevent to both parts. Other points of overlap are the second theorem of 4 in part A and the digression of 2 in part B.

We assemble below definitions and remarks, mostly of a general nature, in order to clarify notation and aquaint the reader with the various facts which will be used without reference later on.

A coalgebra over a field k is a triple (C,μ,ε) where C is a vector space over k and $\mu : C \to C \otimes_k C$ and $\varepsilon : C \to k$ are k maps satisfying

$$(1 \otimes \mu)\mu = (\mu \otimes 1)\mu \qquad \text{and} \qquad (\varepsilon \otimes 1)\mu = (1 \otimes \varepsilon)\mu = 1_C.$$

We often refer to a coalgebra (C,μ,ε) simply as C. The most important coalgebras occurring in this thesis are Hopf algebras.

A Hopf algebra (C,μ,ε) over k is a coalgebra over k such that C is a k algebra, μ and ε are k algebra maps and there is a k map $S : C \to C$ which has the property that

(1) for any $c \in C$, $\sum_i S(c_i)c_i' = \sum_i c_i S(c_i') = \varepsilon(c)1$

where $\mu(c) = \sum_i c_i \otimes c_i'$. S is called the antipode of (C,μ,ε) and is the unique k map satisfying (1). A Hopf algebra (C,μ,ε) is called commutative if C is a commutative k algebra. Hopf algebras occurring in this thesis are commutative exept where an indication to the contrary is given. The most interesting Hopf algebràs from our point of view are those arising from algebraic groups. Algebraic groups should be understood to be affine throughout.

Let K be an algebraically closed field and let $(G,K[G])$ be an algebraic group over K. We regard $K[G]$ as a set of functions on G and $K[G]\otimes K[G]$ as a set of functions on $G\times G$. $K[G]$ is naturally a Hopf algebra with structure maps μ,ε given by $\mu(a)(x,y) = a(xy)$ for any $a \in K[G]$, $x,y \in G$ and $\varepsilon(b) = b(1)$ for any $b \in K[G]$.

Let (C,μ,ε) be a Hopf algebra over k, a Hopf ideal is an ideal I of C such that $\mu(I) \leq I \otimes C + C \otimes I$, $\varepsilon(I) = 0$ and $S(I) \leq I$. It is clear how C/I may be given the structure of a Hopf algebra. A Hopf ideal I is normal if

$m_{13}(1 \otimes 1 \otimes S)(\mu\otimes 1)\mu(I) \leq C \otimes I$ where $m_{13}: C \otimes C \otimes C \to C \otimes C$ is the k map such that $m_{13}(a \otimes b \otimes c) = ac \otimes b$ for any $a,b,c \in C$.

If $(G, K[G])$ is an algebraic group and H is a closed subgroup then $I = \{a \in K[G] \mid a(h) = 0$ for all $h \in H\}$ is a Hopf ideal. If H is in addition normal then I is a normal Hopf ideal. However there are other Hopf ideals of $K[G]$ which do not arise in this way, in general.

Let (C, μ, ε) be a coalgebra over k. A left C comodule is a pair (V, τ) where V is a k space and $\tau : V \to V \otimes_k C$ is a k map satisfying

$$(\tau \otimes 1)\tau = (1 \otimes \mu)\tau \quad \text{and} \quad (1 \otimes \varepsilon)\tau = 1_V.$$

We often denote a C comodule (V, τ) simply by V. $\underset{\sim}{M}(C)$ denotes the category of left comodules (and comodule maps).

We may similarly define right comodules.

Let $V = (V, \tau)$ be a left C comodule and X a k space, we use $(X) \otimes V$ to denote the left comodule $(X \otimes V, 1 \otimes \tau)$. Let $\{v_i\}_{i \in I}$ be a basis of V, the elements t_{ij} determined by

$\tau(v_i) = \sum_{j \in I} v_j \otimes t_{ji}$ are called coefficient functions

and we put $cf(V) = K\ \mathrm{span}\{t_{ij}\}_{i,j \in I}$. Now suppose V is finite dimensional. Define elements f_i of $V^* = \mathrm{Hom}_k(V, k)$ by $f_i(v_j) = \delta_{ij}$ for $i,j \in I$. We may give V^* the structure of a right comodule (V^*, τ_R^*) by defining

$$\tau_R^*(f_j) = \sum_{i \in I} t_{ji} \otimes f_i.$$ If (C, μ, ε) is a Hopf algebra having antipode S then we may give V the structure of the dual left comodule (V, τ_L^*) where $\tau_L^*(v_i) = \sum_{j \in I} v_j \otimes S(t_{ij})$ for any $i \in I$.

From a coalgebra (C,μ,ε) over k we obtain an algebra $C^* = \mathrm{Hom}_k(C,k)$ where multiplication is defined by

(2) $\gamma_1\gamma_2(c) = \sum_i \gamma_1(c_i)\gamma_2(c_i')$ for $\gamma_1, \gamma_2 \in C^*$ and $c \in C$

with $\mu(c) = \sum_i c_i \otimes c_i'$. A left C comodule (V,τ) may be regarded as a left C^* module by defining

$\gamma \circ v = \sum_i v_i\gamma(c_i)$ for $\gamma \in C^*$, $v \in V$ and $\tau(v) = \sum_i v_i \otimes c_i$.

We may refer to a C^* module arising in this way as a rational C^* module.

Similarly we may regard a right C comodule as a right C^* module, in particular C becomes a C^* bimodule in this way.

Let (V,τ) and (V',τ') be left C comodules. A C morphism, or comodule map, $\theta : V \to V'$ is a k map such that

$\tau'\theta = (\theta \otimes 1)\tau$. We write $\mathrm{Hom}_C(V,V')$ for the k space of all C morphisms. $\mathrm{Hom}_C(V,V')$ may be naturally identified with $\mathrm{Hom}_{C^*}(V,V')$.

If $C = K[G]$ for an algebraic group G over K, we may identify G with a subset of C^* by $g \to \varepsilon_g$ where

$\varepsilon_g(a) = a(g)$ for any $g \in G$, $a \in K[G]$. By a rational KG module we mean a rational $K[G]^*$ module. Thus K[G] is a rational KG bimodule. If V and V' are rational G modules we may, and frequently do, identify $\mathrm{Hom}_{KG}(V,V')$ and $\mathrm{Hom}_{K[G]}(V,V')$.

Further details of much of the above material may be found in [6] and [12] of the references for part A.

F Complements and injective comodules

Introduction

In this part we investigate U.L. Hopf algebras (see 1.3). Our aim is to understand, in good cases, the injective indecomposable comodules of a Hopf algebra in terms of a finite dimensional subcomodule, an F complement.

In section 1 we give examples of U.L. algebras and establish some general properties. In section 2 we follow a well worn path to reach a theory of restriction and induction for Hopf algebras. This theory is used in section 3 where we concern ourselves with U.L. algebras arising from algebraic Chevalley groups. We give a proof of a conjecture of Humphreys on the nature of the injective comodules of a simply connected algebraic Chevalley group in characteristic $p > 0$. We also calculate fairly explicitly some related F complements. In section 4 we make some general remarks on the nature of the blocks of a simply connected algebraic Chevalley group in characteristic $p > 0$.

1 Complements and U.L. algebras

1.1

Notation

Throughout part A, K denotes a perfect field of non zero characteristic p. In this section we simply write $\otimes$ for $\otimes_K$ and dim for $\dim_K$.

Let A be a commutative algebra over K, I an ideal of A, V, W subspaces of A and $n \in \mathbb{N}$.

Define

$$I^{[p^n]} = \sum_{a \in I} Aa^{p^n} \quad , \quad V^{(p^n)} = \{ v^{p^n} \mid v \in V \}$$

(a subspace) and

$$V.W = K \text{ span } \{ vw \mid v \in V, w \in W \}.$$

Put $V_1 = V$, $V_n = V.V_{n-1}{}^{(p)}$ so that V_n is spanned by the elements of the form $v_1 \, v_2{}^p \ldots . v_n{}^{p^{n-1}}$, all $v_i \in V$.

By an augmentation algebra we mean a pair (A, ε), where A is a commutative algebra over a field k, say, and $\varepsilon : A \to k$ is a k algebra map. We put $\mathfrak{m} = \ker \varepsilon$.

Definition A uniformly layered (U.L) algebra is a Noetherian augmentation algebra (A, ε) over K such that

$$\dim A/\mathfrak{m}^{[p^n]} = p^{nl} \text{ for each } n \in \mathbb{N} \text{ where } l = \dim \mathfrak{m}/\mathfrak{m}^2 .$$

<u>Remarks</u>

1. For any Noetherian augmentation algebra A over K, we have (1) $\dim A/\mathfrak{M}[p^n] \leq p^{nl}$, where $l = \dim \mathfrak{M}/\mathfrak{M}^2$ (see proof of the lemma of §2 in [11]). Further if $\{a_i + \mathfrak{M}^2\}_{i=1,2\ldots l}$ is a basis of $\mathfrak{M}/\mathfrak{M}^2$, we have equality in (1) if and only if $a_1^{p^n-1} a_2^{p^n-1} \ldots a_l^{p^n-1} \notin \mathfrak{M}^{[p^n]}$.

2. If (A,ε) and (A',ε') are U.L. algebras finitely generated over K then $(A \otimes A', \varepsilon \otimes \varepsilon')$ is U.L.

3. Let (A,ε) be an augmentation algebra over K such that A is an integral domain, we may form an augmentation algebra $(A_{\mathfrak{M}}, \varepsilon_{\mathfrak{M}})$ where $\varepsilon_{\mathfrak{M}}(\frac{a}{b}) = \frac{\varepsilon(a)}{\varepsilon(b)}$ for $a,b \in A$, $b \notin \mathfrak{M}$. If A is Noetherian then (A,ε) is U.L. if and only if $(A_{\mathfrak{M}}, \varepsilon_{\mathfrak{M}})$ is U.L.

<u>Examples</u> (a) $A = K[X_1, X_2, \ldots X_l]$, $\varepsilon : A \to K$ any K algebra map. By a change of coordinates, if necessary, we may assume that $\varepsilon(X_i) = 0$ for all $1 \leq i \leq l$. So

$$\mathfrak{M} = \sum_{i=1}^{l} AX_i, \quad \mathfrak{M}^{[p^n]} = \sum_{i=1}^{l} AX_i^{p^n} \quad \text{and } A/\mathfrak{M}[p^n]$$

may be identified with the polynomials in $\{X_1, \ldots X_l\}$ of degree less than p^n in each variable.

In particular if K is algebraically closed and U is a connected unipotent algebraic group over K, then $K[U]$ is a free polynomial ring by [O, (3.3)] . By the above, if $g \in U$

and $\varepsilon_g : K[U] \to K$ is evaluation at g then $(K[U], \varepsilon_g)$ is a U.L. algebra.

(b) Let H be a torus over K, algebraically closed, $\varepsilon : K[H] \to K$ evaluation at 1.

$$K[H] \cong K\left[T_1, \ldots T_n, \frac{1}{T_1}, \frac{1}{T_2}, \ldots \frac{1}{T_n}\right] \text{ for some } n, \ \varepsilon(T_i) = 1$$

for all i, so by example (a) and remark 3, $(K[H], \varepsilon)$ is U.L.

(c) Let K be algebraically closed, B any soluble connected algebraic group over K and $\varepsilon: K[B] \to K$ evaluation at 1. Then $B \cong H.U$, the semidirect product of a maximal torus H and U the subgroup of B consisting of the unipotent elements.

$K[B] \cong K[H] \otimes K[U]$ so $(K[B], \varepsilon)$ is U.L. by (a), (b) and remark 2.

(d) Let G be a connected semisimple algebraic group over K algebraically closed and ε evaluation at 1. $(K[G], \varepsilon)$ is U.L. This is proved in [1, Proposition of §2]. It also follows from Chevalley's Theorem [4, Proposition 1], (a), (b) together with remarks 2 and 3.

1.2

<u>Definition</u> A complement of a U.L. algebra (A, ε) is a subspace V of A such that $1 \in V$ and $A = V \oplus \mathfrak{m}^{[p]}$.

The importance of complements lies in the following.

<u>Theorem</u> If V is a complement for a U.L. algebra $(A,\mathcal{E})$ then

$$A = V_n + \mathfrak{M}^{[p^n]} \quad \text{for any } n \in \mathbb{N}\,.$$

<u>Proof</u> True for $n=1$ by definition of complement. Assume that $n > 1$ and the theorem holds for $n-1$.

Put $X = V_n + \mathfrak{M}^{[p^n]}$ and $T_m = \underbrace{(\mathfrak{M}\cap V)^{(p^{n-1})}\cdots(\mathfrak{M}\cap V)^{(p^{n-1})}}_{m \text{ times}}$.

We will prove by induction that, for all m, $A = X + AT_m$ (A).

Since $A/\mathfrak{M}^{[p^n]}$ is local, for some m', $T_{m'} \subseteq \mathfrak{M}^{[p^n]}$ so it will follow from (A) that $A = X$ as required.

$\mathfrak{M} = \mathfrak{M}\cap V + \mathfrak{M}^{[p]}$, by the modular law, so

$$\mathfrak{M}^{[p^{n-1}]} = A(\mathfrak{M}\cap V)^{(p^{n-1})} + \mathfrak{M}^{[p^n]} \qquad (1).$$

$$A = V_{n-1} + \mathfrak{M}^{[p^{n-1}]} = V_{n-1} + A(\mathfrak{M}\cap V)^{(p^{n-1})} + \mathfrak{M}^{[p^n]}$$

$$= X + A(\mathfrak{M}\cap V)^{(p^{n-1})}$$

$$(1 \in V \text{ implies } V_1 \subseteq V_2 \subseteq \ldots\ldots)\,.$$

Hence (A) is true for $m=1$.

Note $A = V + \mathfrak{M}^{[p]}$ gives $A^{(p^{n-1})} \subseteq V^{(p^{n-1})} + \mathfrak{M}^{[p^n]}$

and so $A^{(p^{n-1})}V_{n-1} \subseteq X$ (2).

Suppose $A = X + AT_m$ then

$$A = X + V_{n-1}T_m + \mathfrak{M}^{[p^{n-1}]}T_m$$

$$= X + A(\mathfrak{M}\cap V)^{(p^{n-1})}T_m = X + AT_{m+1}$$

using (1), (2) and the hypothesis $A = V_{n-1} + \mathfrak{M}^{[p^{n-1}]}$.

Hence (A) is true by induction so

$$A = X = V_n + \mathfrak{m}^{[p^n]} \quad (3).$$

Since A is U.L. , $\dim A/\mathfrak{m}^{[p^n]} = p^{n\dim \mathfrak{m}/\mathfrak{m}^2}$

$= (\dim A/\mathfrak{m}^{[p]})^n = (\dim V)^n.$

There is a K epimorphism

$$V \otimes V^{(p)} \otimes \ldots\ldots \otimes V^{(p^{n-1})} \to VV^{(p)}\ldots V^{(p^{n-1})} = V_n \quad (4)$$

given by $v_1 \otimes v_2{}^p \otimes \ldots \otimes v_n{}^{p^{n-1}} \to v_1 v_2{}^p \ldots\ldots v_n{}^{p^{n-1}}$.

Hence $\dim V_n \leqslant (\dim V)^n$. This shows that the sum (3) is direct and proves the theorem.

<u>Remark</u> As a corollary to the above argument we obtain

$\dim V_n = (\dim V)^n$, $\dim V^{(p^n)} = \dim V$ for any $n \geqslant 0$ and the map of (4) is an isomorphism.

<u>Definition</u> For $W \leqslant A$, A a commutative K algebra put $W_\infty = \bigcup_{n>0} W_n$. Say a complement V of a U.L. algebra $(A, \mathcal{E})$ is strong if $V_\infty = A$.

<u>Examples</u> (a) $A = K[X]$, $\mathcal{E}(X) = 0$. Put $V = K + KX + \ldots + KX^{p-1}$. $\mathfrak{m}^{[p]} = AX^p$ so V is a complement. $V_\infty = A$.

(b) $A = K[Y, \frac{1}{Y}]$, $\mathcal{E}(Y) = 1$.

(i) $V = K + KY + \ldots\ldots + KY^{p-1}$ is a complement, but $V_\infty = K[Y] \neq A$, V is not strong.

(ii) For $p > 2$, $V = K$ span $\left\{ \frac{1}{Y^{\frac{p-1}{2}}}, \frac{1}{Y^{\frac{p-3}{2}}}, \ldots\ldots, Y^{\frac{p-1}{2}} \right\}$

is a strong complement.

We conclude 1.2 with a proposition and its limiting version. Let $(A,\mathcal{E})$ be a U.L. algebra over K, V a subspace of A such that $V \cap \mathfrak{M}^{[p]} = 0$. Let $V = V(0) \oplus V(1) \oplus \ldots\ldots \oplus V(n)$ be any K space decomposition of V. Put T_m = the set of functions $\{0,1,\ldots m\} \to \{0,1,\ldots n\}$. For $f \in T_m$ define

$$V(f) = V(f(0)).V(f(1))^{(p)} \ldots\ldots V(f(m))^{(p^m)}.$$

<u>Proposition</u> (a) With the above hypotheses

$$V_{m+1} = \sum_{f \in T_m} \oplus V(f).$$

<u>Proof</u> V_{m+1} is spanned by elements of the form $v_{i_0} v_{i_1}{}^{p} \ldots\ldots v_{i_m}{}^{p^m}$ where $v_{i_r} \in V(i_r)$, $0 \leqslant i_r \leqslant n$.

But $v_{i_0} v_{i_1}{}^{p} \ldots v_{i_m}{}^{p^m} \in V(f)$, where $f(r) = i_r$ for $0 \leqslant r \leqslant m$.

Hence $V_{m+1} = \sum_{f \in T_m} V(f)$ (1).

Let W be a complement for A containing V. By the remark following the theorem we have $\dim W^{(p^r)} = \dim W$ for $r > 0$ and the natural map $W \otimes W^{(p)} \otimes \ldots\ldots \otimes W^{(p^m)} \to W_{m+1}$ is an isomorphism of K spaces. Hence $V \leqslant W$ gives $\dim V^{(p^r)} = \dim V$ and $\dim V(s)^{(p^r)} = \dim V(s)$ for $r \geqslant 0$, $0 \leqslant s \leqslant n$. Also the natural maps $V \otimes V^{(p)} \otimes \ldots \otimes V^{(p^m)} \to V_{m+1}$ and

$V(f(0)) \otimes V(f(1))^{(p)} \otimes \ldots \otimes V(f(m))^{(p^m)} \to V(f)$

are isomorphisms of K spaces for any $f \in T_m$.

Thus $\dim V_{m+1} = (\dim V)^{m+1}$ (2) and

$\dim V(f) = \dim V(f(0)).\dim V(f(1))\ldots\dim V(f(m))$ (3).

Using (2) and (3) one can check that $\sum_{f \in T_m} \dim V(f) = \dim V_{m+1}$.
It follows that the sum in (1) is direct as required.

Assume further that $1 \in V(0)$. Put S = the set of functions $\mathbb{Z}^+ \to \{0,1,\ldots n\}$ of finite support. For $f \in S$ put

$\alpha(f) = \max_{f(r) \neq 0} \{r\}$. For $m \in \mathbb{Z}^+$, $f \in S$ define

$V(f,m) = V(f(0)).V(f1))^{(p)} \ldots\ldots V(f(m))^{(p^m)}$ and

$V(f) = \bigcup_{m \geqslant \alpha(f)} V(f,m)$.

Proposition (b) With the above hypotheses

$$V_\infty = \sum_{f \in S}^{\oplus} V(f).$$

Proof Put $S_m = \{f \in S \mid \alpha(f) \leqslant m\}$. We see that $V_{m+1} \subseteq \sum_{f \in S_m} V(f)$. Since $V_\infty = \bigcup_{m \geqslant 1} V_m$ this shows that

$$V_\infty \subseteq \sum_{f \in S} V(f).$$

$V(f) \subseteq V_\infty$ for any $f \in S$ is clear so $V_\infty = \sum_{f\in S} V(f)$.

Now suppose $\sum_{i=1}^{l} v_{f_i} = 0$, each $v_{f_i} \in V(f_i)$ and $f_1, \ldots, f_l$ distinct elements of S. Pick $m \in \mathbb{Z}^+$ such that $m \geq (f_i)$ for all i and $v_{f_i} \in V(f_i, m)$ for all i. Then $f_1, \ldots f_l$ are distinct elements of S_m and the directness of the sum in proposition (a) gives each $v_{f_i} = 0$.

<u>1.3</u> We shall be most interested in the augmentation algebra (R,ε) arising from a Hopf algebra (R,μ,ε) over K.

<u>Definition</u> Let (R,μ,ε) be a U.L. Hopf algebra over K (i.e. (R,) is U.L.). An F complement is a complement for (R,ε) which is also a left subcomodule of R. That is, an F complement is a subspace V of R such that

$$1 \in V, \quad \mu(V) \leqslant V \otimes R \text{ and } V \oplus \mathfrak{M}^{[p]} = R.$$

<u>Remark</u> One can see that with respect to the appropriate Hopf structures the complements of 1.2(a), 1.2(b) (i) and 1.2(b) (ii) are F complements.

<u>A Criterion for an F complement</u>

What follows is largely a reworking of [2] in a Hopf algebra context. Let (R,μ,ε) be a Hopf algebra over K. Put $T(R) = \{ Y \in R^* \mid Y(rs) = \varepsilon(r)Y(s) + Y(r)\varepsilon(s)$ for all $r,s \in R\}$. T(R) is a Lie algebra with Lie product

$[\gamma_1, \gamma_2] = \gamma_1\gamma_2 - \gamma_2\gamma_1$ for $\gamma_1, \gamma_2 \in T(R)$, multiplication given by formula 2 of General Remarks. The Lie algebra $T(R)$ is restricted with p operation $\gamma^{[p]} = \gamma^p$ for γ in $T(R)$.

Now suppose $\mathfrak{M}^{[p]} = 0$ and R is finite dimensional. Put $L(N) = K$ span $\{ Y_1 Y_2 \ldots Y_t \mid Y_i \in T(R)$ and $t \leq N\}$.

$$= K + T(R) + T(R)^2 + \ldots + T(R)^N.$$

Lemma 1 If $Y \in L(N)$, $l > N$ then $Y \circ \mathfrak{M}^l \subseteq \mathfrak{M}^{l-N}$.

Proof Clearly it is enough to prove the result for $N = 1$. If $Y \in T(R)$, $m_1, \ldots m_l \in \mathfrak{M}$, $r \in R$ then

$$Y\circ(rm_1 \ldots m_l) = (Y\circ(m_1 m_2 \ldots m_{l-1} r))m_l + (rm_1 \ldots m_{l-1}) Y\circ m_l.$$

It is easy to see that the above equation may be used to give a proof of the lemma by induction on l.

Corollary If $Y \in L(N)$, $Y(\mathfrak{M}^{N+1}) = 0$.

Let $\{X_1, \ldots X_n\}$ be a basis for $T(R)$, by 13.2.3 of [14] $T(R)$ generates the restricted enveloping algebra of $T(R)$, it follows that $\dim R = p^n$ and that

$$\mathcal{B} = \{ X_1^{\alpha_1} \ldots X_n^{\alpha_n} \mid 0 \leq \alpha_i \leq p-1\} \text{ is a basis for } R^*.$$

We refer to an element of $\mathcal{B}$ as a monomial. Let $X_0 = X_1^{p-1} \ldots X_n^{p-1}$ and define $\varphi : R^* \to K$ by

$\varphi(D)$ = coefficient of X_0 in an expression of D as a sum of monomials, for $D \in R^*$.

Define degree of $\sum_{\substack{\underline{\alpha}=(\alpha_1,\ldots,\alpha_n)\\ 0\leq\alpha_i\leq p-1}} \lambda_{\underline{\alpha}} X_1^{\alpha_1}\ldots X_n^{\alpha_n}$

$= \max \{ \sum_i \alpha_i \mid \lambda_{\underline{\alpha}} \neq 0\}$.

For $D \in R^*$ write $\partial(D)$ = degree of D. We define $\partial(0) = -1$.

The following facts may easily be verified: if $D_1, D_2 \in R^*$, $D_1 \neq 0 \neq D_2$ then

$$\partial(D_1D_2) \leq \partial(D_1) + \partial(D_2) \text{ and } \partial([D_1,D_2]) < \partial(D_1) + \partial(D_2) \quad (1)$$

where $[D_1,D_2] = D_1D_2 - D_2D_1$.

<u>Lemma 2</u> If $\varphi(X_1^{\alpha_1}\ldots X_n^{\alpha_n} X_1^{\beta_1}\ldots X_n^{\beta_n}) \neq 0$ and

$\sum_i \alpha_i + \sum_i \beta_i \leq n(p-1)$ then $\alpha_i + \beta_i = p-1$

for $1 \leq i \leq n$.

<u>Proof</u> Clearly by (1) we must have $\sum_i \alpha_i + \sum_i \beta_i = n(p-1)$.

If $(\alpha_1,\ldots,\alpha_n) = (0,0,\ldots 0)$ then it is obvious. Now we prove the lemma by induction on the number of nonzero α_i's.

Suppose $\alpha_r \neq 0$, then

$$X_1^{\alpha_1}\ldots X_r^{\alpha_r}\ldots X_n^{\alpha_n} X_1^{\beta_1}\ldots X_n^{\beta_n} =$$

$$X_1^{\alpha_1}\ldots X_{r-1}^{\alpha_{r-1}} X_{r+1}^{\alpha_{r+1}}\ldots X_n^{\alpha_n} X_1^{\beta_1}\ldots X_r^{\alpha_r+\beta_r}\ldots X_n^{\beta_n}$$

$$+ X_1^{\alpha_1}\ldots X_{r-1}^{\alpha_{r-1}} [X_r^{\alpha_r}, X_{r+1}^{\alpha_{r+1}}\ldots X_n^{\alpha_n} X_1^{\beta_1}\ldots X_r^{\beta_r}] X_{r+1}^{\beta_{r+1}}\ldots X_n^{\beta_n}$$

$$= X_1^{\alpha_1}\ldots X_{r-1}^{\alpha_{r-1}} X_{r+1}^{\alpha_{r+1}}\ldots X_n^{\alpha_n} X_1^{\beta_1}\ldots X_r^{\alpha_r+\beta_r}\ldots X_n^{\beta_n}$$

+ terms of smaller degree, by (1).

Hence $\varphi(X_1^{\alpha_1}\ldots X_{r-1}^{\alpha_{r-1}}X_{r+1}^{\alpha_{r+1}}\ldots X_n^{\alpha_n}X_1^{\beta_1}\ldots X_r^{\alpha_r+\beta_r}\ldots X_n^{\beta_n}) \neq 0$

and so by induction we get $\alpha_i + \beta_i = p-1$, $i \neq r$

and $0 + (\alpha_r + \beta_r) = p-1$.

Thus for all i, $\alpha_i + \beta_i = p-1$.

<u>Proposition</u> If $\{a_i + \mathfrak{M}^2\}_{i=1,\ldots n}$ is a basis for $\mathfrak{M}/\mathfrak{M}^2$ then R is a free cyclic left R^* module generated by $a_1^{p-1}a_2^{p-1}\ldots a_n^{p-1}$.

<u>Proof</u> It is easy to check that $\{a_1,\ldots a_n\}$ generate R as a K algebra. $\mathfrak{M}^{[p]} = 0$ and so each $a_i^p = 0$. Now since $\dim R = p^n$ we get that $\{a_1^{l_1}\ldots\ldots a_n^{l_n} \mid 0 \leq l_i \leq p-1\}$ is a basis for R. By lemma 1 for $0 \leq \alpha_i \leq p-1$

(2) $(X_1^{\alpha_1}\ldots X_n^{\alpha_n})(a_0) = 0$ unless $\alpha_1 = \alpha_2 = \ldots = \alpha_n = p-1$

where $a_0 = a_1^{p-1}\ldots a_n^{p-1}$. Hence since

$\{X_1^{\ 1}X_2^{\ 2}\ldots X_n^{\ n} \mid 0 \leq \alpha_i \leq p-1\}$ is a basis for R^*,

$X_1^{p-1}\ldots X_n^{p-1}(a_0) \neq 0$. Thus

$$(3) \quad X_1^{p-1}\ldots X_n^{p-1} \circ a_0 \notin \mathfrak{M} .$$

Now suppose $D \circ a_0 = 0$ for some $0 \neq D \in R^*$,

$$D = \sum_{\underline{\alpha}} \lambda_{\underline{\alpha}} X_1^{\alpha_1}\ldots X_n^{\alpha_n}$$, say. Suppose that

$\partial(D) = \sum_i \beta_i$ for $\lambda_{\underline{\beta}} \neq 0$. Then

$(X_1^{p-1-\beta_1}\ldots\ldots X_n^{p-1-\beta_n} D) \circ a_0 = 0$. Now by the above lemma the coefficient of $X_1^{p-1}\ldots X_n^{p-1}$ in an expression of $X_1^{p-1-\beta_1}\ldots X_n^{p-1-\beta_n} D$ as a linear combination of monomials is λ_{β} and so by (2) we have

$\lambda_{\beta}(X_1^{p-1}\ldots X_n^{p-1})(a_0) = 0$. But $\lambda_{\beta} \neq 0$ so we have a contradiction to (3). Thus $D = 0$ and so the left annihilator of $a_0 = a_1^{p-1}\ldots a_n^{p-1}$ is 0. Hence

$$\dim R^* \circ a_0 = p^{\dim T(R)} = \dim R$$

and so $R^* \circ a_0 = R$ and we are done.

<u>Corollary</u> R^* is a Frobenius algebra.

<u>Proof</u> The natural map $R \to ({R^*}_{R^*})^*$ is a left R^* module isomorphism since $\dim R < \infty$. But the proposition shows $R \cong ({}_{R^*}R^*)$ as left R^* modules thus ${}_{R^*}R^* \cong ({R^*}_{R^*})^*$, i.e. R^* is Frobenius.

We no longer insist that $\mathfrak{M}^{[p]} = 0$.

<u>Lemma 3</u> Suppose R is a finitely generated Hopf algebra over K and $\{a_i + \mathfrak{M}^2\}_{i=1,\ldots n}$ a K basis for $\mathfrak{M}/\mathfrak{M}^2$, $a_0 = a_1^{p-1}\ldots a_n^{p-1} + \mathfrak{M}^{[p]}$ and $\pi : R \to R/\mathfrak{M}^{[p]}$ the natural map. If $\pi(r) = a_0$ then $L_R(r) + \mathfrak{M}^{[p]} = R$ where $L_R(r)$ is the left subcomodule of R generated by r.

<u>Proof</u> Work mod $\mathfrak{M}^{[p]}$ and apply the proposition.

<u>Criterion</u> In the situation of the above lemma, if R is U.L., $\dim L_R(r) \leqslant p^{\dim \mathfrak{M}/\mathfrak{M}^2}$ and $1 \in L_R(r)$ then $L_R(r)$ is an F complement for R.

<u>Examples</u> (a) Winter's F complement

$R = K[SL_2(K)] = K[A,B,C,D \mid AD - BC = 1]$ where K is algebraically closed and the functions A,B,C,D are defined by

$$g = \begin{pmatrix} A(g) & B(g) \\ C(g) & D(g) \end{pmatrix} \quad \text{for any } g \in G = SL_2(K).$$

Put $V = K \text{ span } \{ A^\alpha B^\beta C^\gamma D^\delta \mid 0 \leqslant \alpha+\beta \leqslant p-1 \text{ and } 0 \leqslant \gamma+\delta \leqslant p-1 \}$

One can easily check that V is a left rational G module.

$\mathfrak{M} = R(A-1) + RB + RC + R(D-1)$ and

$\{A - 1 + \mathfrak{M}^2,\ B + \mathfrak{M}^2,\ C + \mathfrak{M}^2\}$ is a basis of $\mathfrak{M}/\mathfrak{M}^2$.

Let $\equiv$ denote congruence mod $\mathfrak{M}^{[p]}$. Note $A^p - 1 \equiv (A-1)^p$.

$$(A-1)^{p-1}B^{p-1}C^{p-1} = (A-1)^{p-1}(AD-1)^{p-1}$$

$$= \left(\sum_{r=0}^{p-1} A^r\right)\left(\sum_{s=0}^{p-1} A^s D^s\right) \equiv \sum_{r,s=0}^{p-1} A^r D^s \in V \quad (1).$$

Clearly $1 = AD - BC \in V$ and using this relation

$$V = K \text{ span } \{ A^\alpha B^\beta C^\gamma D^\delta \mid 0 \leqslant \alpha+\beta,\ \gamma+\delta \leqslant p-1 \text{ and } \alpha+\beta = p-1 \text{ or } \gamma+\delta = p-1 \}.$$

Hence $\dim_K V \leqslant p(1 + 2 + \dots + p) + p(1 + 2 + \dots + p-1)$

$$= \tfrac{1}{2}p^2(p+1) + \tfrac{1}{2}p^2(p-1) = p^3 \quad (2).$$

Thus by (1), (2) and our criterion V is an F complement.

(b) $R = K[G]$, $G = U(3,K)$ the group of upper triangular unipotent 3 by 3 matrices.

$R = K[X,Y,Z]$ where X,Y,Z are defined by

$$g = \begin{pmatrix} 1 & X(g) & Y(g) \\ 0 & 1 & Z(g) \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for any } g \in G.$$

Define A and B by $A(g) = Y(g^{-1})$ and $B(g) = Z(g^{-1})$ for $g \in G$. So $A = XZ - Y$, $B = -Z$.

Put $V = K \text{ span } \{ X^{\alpha}Y^{\beta}A^{\gamma}B^{\delta} \mid 0 \leqslant \alpha+\beta , \gamma+\delta \leqslant p-1 \}$.

Check V is a left G submodule of R. The relation

$BX + A + Y = 0$ gives

$V = K \text{ span } \{ X^{\alpha}Y^{\beta}A^{\gamma}B^{\delta} \mid 0 \leqslant \alpha+\beta , \gamma+\delta \leqslant p-1 \text{ and } \min\{\alpha,\delta\} = 0 \}$.

Hence $\dim V \leqslant p^3$ as in example (a) (1).

$\mathfrak{M} = RX + RY + RZ$, and $\{X + \mathfrak{M}^2, Y + \mathfrak{M}^2, Z + \mathfrak{M}^2\}$ is a basis for $\mathfrak{M}/\mathfrak{M}^2$. As before $\equiv$ denotes congruence mod $\mathfrak{M}^{[p]}$.

We have $X^{p-1}Y^{p-1}Z^{p-1} = Y^{p-1}(A+Y)^{p-1} \equiv A^{p-1}Y^{p-1} \in V$.

$1 \in V$ so by (1) and the criterion V is an F complement.

<u>Remark</u> 1 The V of example (b) is a special case of the F complement described in 3.3.

2 It is not difficult to do examples (a) and (b) without appeal to the criterion, see [5].

2 Restriction and Induction

2.1

In this section we develop a theory of restriction and induction for coalgebras and Hopf algebras. This closely parallels, and may be regarded as a generalisation of, the set up in [10]. The theory will be used in the next section to show that certain comodules are injective.

<u>Notation</u>

Throughout this section comodule means left comodule. Let $(C, \mu, \mathcal{E})$ be a coalgebra over a field k. Let (V, Υ) be a comodule, if X is any k space $(X) \otimes V$ denotes the C comodule $(X \otimes V, 1_X \otimes \Upsilon)$ as in [6, 1.2f]. If $(C, \mu, \mathcal{E})$ is a Hopf algebra we denote by k the trivial one dimensional comodule.

<u>φ_o and φ^o</u> Let $(A, \mu_A, \mathcal{E}_A)$ and $(B, \mu_B, \mathcal{E}_B)$ be coalgebras over k and let $\varphi : A \to B$ be a coalgebra map (see [12, 1.3]).
φ determines functors $\varphi_o : \underset{\sim}{\mathcal{M}}(A) \to \underset{\sim}{\mathcal{M}}(B)$ the φ restriction functor and $\varphi^o : \underset{\sim}{\mathcal{M}}(B) \to \underset{\sim}{\mathcal{M}}(A)$ the φ induction functor.

If (V, Υ) is an A comodule then $\varphi_o(V)$ is defined to be the B comodule $(V, (1 \otimes \varphi)\Upsilon)$. If (V', Υ') is also an A comodule and $f \in \mathrm{Hom}_A(V, V')$ then write $\varphi_o(f)$ for the k map f regarded as an element of $\mathrm{Hom}_B(\varphi_o(V), \varphi_o(V'))$. It is easy to see that φ_o is a covariant functor.

Let (W,σ) be a B comodule, $\varphi^{o}(W)$ is the A subcomodule of $(W)\otimes A$ consisting of the elements $\sum_i w_i \otimes a_i$, $w_i \in W$, $a_i \in A$ such that

$$(\sigma \otimes 1)\left(\sum_i w_i \otimes a_i\right) = (1 \otimes (\varphi \otimes 1)\mu_A)\left(\sum_i w_i \otimes a_i\right).$$

This is a subcomodule since $\sigma \otimes 1$ and $1 \otimes (\varphi \otimes 1)\mu_A$ may be regarded as A morphisms $: (W)\otimes A \to (W \otimes B)\otimes A$ and $\varphi^{o}(W)$ is the kernel of $\sigma \otimes 1 - (1 \otimes (\varphi \otimes 1)\mu_A)$.

Suppose (W',σ') is also a B comodule and $f \in \mathrm{Hom}_B(W,W')$, we define $\varphi^{o}(f) : \varphi^{o}(W) \to \varphi^{o}(W')$ by $\varphi^{o}(f)\left(\sum_i w_i \otimes a_i\right) = \sum_i f(w_i)\otimes a_i$. One can check

that $\varphi^{o}(f)$ is well defined, an element of $\mathrm{Hom}_A(\varphi^{o}(W), \varphi^{o}(W'))$ and that φ^{o} is a covariant functor.

Define the k map $e : \varphi^{o}(W) \to W$ by $e\left(\sum_i w_i \otimes a_i\right) = \sum_i \varepsilon_A(a_i)w_i$ for $\sum_i w_i \otimes a_i \in \varphi^{o}(W)$.

Using that $1 \otimes (\varphi \otimes 1)\mu_A - \sigma \otimes 1$ is zero on $\varphi^{o}(W)$, one may check that the diagram

$$\begin{tikzcd}
\varphi_o(\varphi^{o}(W)) \arrow[r, "e"] \arrow[d, "(1\otimes 1\otimes \varphi)(1\otimes \mu_A)"'] & W \arrow[d, "\sigma"] \\
\varphi_o(\varphi^{o}(W))\otimes B \arrow[r, "e\otimes 1"'] & W\otimes B
\end{tikzcd}$$

commutes, i.e. e is a B morphism.

Universal Mapping Property

Let $(X,\uparrow)$ be any A comodule, then if $\alpha \in \mathrm{Hom}_B(\varphi_o(X),W)$ there is a unique $\hat{\alpha} \in \mathrm{Hom}_A(X,\varphi^o(W))$ such that $e\hat{\alpha} = \alpha$, i.e. the diagram

$$\begin{tikzcd} X \arrow[rr, "\hat{\alpha}"] \arrow[dr, "\alpha"'] & & \varphi^o(W) \arrow[dl, "e"] \\ & W & \end{tikzcd}$$

commutes.

Proof Define maps $\Delta_1 : \mathrm{Hom}_B(\varphi_o(X),W) \to \mathrm{Hom}_A(X,\varphi^o(W))$ and $\Delta_2 : \mathrm{Hom}_A(X,\varphi^o(W)) \to \mathrm{Hom}_B(\varphi_o(X),W)$ by $\Delta_1(\alpha) = (\alpha \otimes 1)\uparrow$ and $\Delta_2(\theta) = e\theta$ for $\alpha \in \mathrm{Hom}_B(\varphi_o(X), W)$ and $\theta \in \mathrm{Hom}_A(X,\varphi^o(W))$. Check Δ_1, Δ_2 are well defined and are inverse isomorphisms.

Remark If M is an A comodule, $f \in \mathrm{Hom}_B(\varphi_o(M),W)$ and (M,f) has the above universal mapping property possessed by $(\varphi^o(W),e)$ it follows that there exist unique A isomorphisms u and v such that

$$\begin{tikzcd} M \arrow[rr, shift left, "u"] \arrow[dr, "f"'] & & \varphi^o(W) \arrow[ll, shift left, "v"] \arrow[dl, "e"] \\ & W & \end{tikzcd}$$

commutes.

2.2 Restriction and induction have the following properties:

a) φ^o and φ_o commute with taking a direct sum of comodules ;

b) φ^o is left exact, φ_o is exact ;

c) For any A comodule X and any B comodule W

$$\mathrm{Hom}_A(X, \varphi^{o}(W)) \cong \mathrm{Hom}_B(\varphi_o(X), W) \text{ as k spaces (reciprocity)}$$

d) Let $(C, \mu_C, \varepsilon_C)$ be a coalgebra over k and $\psi : B \to C$ a coalgebra map then (i) $(\psi\varphi)_o$ is naturally isomorphic to $\psi_o\varphi_o$ and (ii) $(\psi\varphi)^o$ is naturally isomorphic to $\varphi^o\psi^o$ (transitivity of induction) ;

e) Regard B as a left B comodule, then $\varphi^o(B) \cong A$ as left comodules;

f) If W is an injective in $\underset{\sim}{M}(B)$ then $\varphi^o(W)$ is an injective in $\underset{\sim}{M}(A)$.

Suppose further that A and B are (commutative) Hopf algebras and that φ is a Hopf algebra map. Recall that the tensor product of two comodules for a Hopf algebra has a natural comodule structure. We have the following additional properties :

g) φ_o commutes with taking a tensor product of comodules ;

h) If (V, γ) is an A comodule and $(W, \S)$ is a B comodule then

$\varphi^o(\varphi_o(V) \otimes W) \cong V \otimes \varphi^o(W)$ as A comodules (the adjoint law).

We shall sketch proofs of d) (ii), e) and h).

<u>Proposition</u> d) (ii) $(\psi\varphi)^o$ is naturally isomorphic to $\varphi^o\psi^o$.

<u>Proof</u> Suppose $(W, \S)$ is a C comodule. Let $f_1 : \psi^o(W) \to W$ be the C morphism defined $f_1(\sum_i w_i \otimes b_i) = \sum_i w_i \varepsilon_B(b_i)$. Similarly define a B morphism $f_2 : \varphi^o(\psi^o(W)) \to \psi^o(W)$.

Put $f = f_1 f_2$.

Now if X is any A comodule and $\alpha \in \mathrm{Hom}_C((\psi\varphi)_o(X), W)$ $= \mathrm{Hom}_C(\psi_o\varphi_o(X), W)$ then by the universal mapping property for $(\psi^o(W), f_1)$ there exists a unique $\hat{\alpha} \in \mathrm{Hom}_B(\varphi_o(X), \psi^o(W))$ such that $f_1\hat{\alpha} = \alpha$. By the universal mapping property for $(\varphi^o(\psi^o(W)), f_2)$ there is a unique $\hat{\hat{\alpha}} \in \mathrm{Hom}_A(X, \varphi^o(\psi^o(W)))$ such that $f_2\hat{\hat{\alpha}} = \hat{\alpha}$. Hence $f\hat{\hat{\alpha}} = f_1 f_2\hat{\hat{\alpha}} = \alpha$. By reversing the steps we get that $\hat{\hat{\alpha}}$ is the unique element of $\mathrm{Hom}_A(X, \varphi^o(\psi^o(W)))$ with this property. Thus $(\varphi^o(\psi^o(W)), f)$ has the universal mapping property of 2.1. Hence there exist unique A isomorphisms f_W, g_W such that

$$\begin{tikzcd}
\varphi^o(\psi^o(W)) \arrow[rr, shift left, "f_W"] \arrow[dr, "f"'] & & (\psi\varphi)^o(W) \arrow[ll, shift left, "g_W"] \arrow[dl, "e"] \\
& W &
\end{tikzcd}$$

commutes.

Naturality may be checked by constructing the isomorphisms f_W and g_W explicitly.

<u>Corollary</u> e) If B is regarded as a left comodule then

$$\varphi^o(B) \cong A \text{ as left } A \text{ comodules.}$$

<u>Proof</u> Let k denote the one dimensional k coalgebra (k, μ, ε) defined by $\mu(1) = 1 \otimes 1$, $\varepsilon(1) = 1$. If $(C, \mu_C, \varepsilon_C)$ is any k coalgebra then $\varepsilon_C : C \to k$ is a k coalgebra map and we claim $\varepsilon_C{}^o(k) \cong C$, as left C comodules.

Now $\varepsilon_C{}^o(k) = \{1 \otimes c \mid (\mu \otimes 1)(1 \otimes c) = 1 \otimes (\varepsilon_C \otimes 1)\mu_C(1 \otimes c)\}$

$= \{1 \otimes c \mid c \in C\} \cong C$ as left C comodules.

Now by d) (ii) $\varphi^o(B) \cong \varphi^o(\varepsilon_B{}^o(k)) \cong (\varepsilon_B\varphi)^o(k)$

$\cong \varepsilon_A{}^o(k) \cong A$ as A comodules.

<u>Proposition</u> h) Let (A,μ_A,ε_A), (B,μ_B,ε_B) be Hopf algebras and $\varphi : A \to B$ a Hopf algebra map. Let (V,Υ) be a left A comodule and (W,τ) a left B comodule. Then

$$\varphi^o(\varphi_o(V) \otimes W) \cong V \otimes \varphi^o(W) \text{ as } A \text{ comodules.}$$

<u>Proof</u> Let $\lambda_1 : \varphi_o(V \otimes \varphi^o(W)) \to \varphi_o(V) \otimes W$ be the B map given by $\lambda_1(v \otimes \sum_i w_i \otimes a_i) = v \otimes \sum_i \varepsilon_A(a_i)w_i$ for $v \in V$, $\sum_i w_i \otimes a_i \in \varphi^o(W)$. Let Λ_1 be the A morphism

$$V \otimes \varphi^o(W) \to \varphi^o(\varphi_o(V) \otimes W) \text{ corresponding to } \lambda_1,$$

given by the universal mapping property. Λ_1 satisfies $\Lambda_1(v \otimes \sum_i w_i \otimes a_i) = \sum_{i,l} v_l \otimes w_i \otimes f_l a_i$ for $v \in V$, $\sum_i w_i \otimes a_i \in \varphi^o(W)$ and $\Upsilon(v) = \sum_l v_l \otimes f_l$.

Let S be the antipode of A. Define $\Lambda_2 : \varphi^o(\varphi_o(V) \otimes W) \to V \otimes \varphi^o(W)$ to be the k map such that

$$\Lambda_2(\sum_{i,j} v_i \otimes w_j \otimes a_{ij}) = \sum_{i,j,l} v_{il} \otimes w_j \otimes S(f_{il})a_{ij}$$

where $v_i \in V$, $w_j \in W$, $a_{ij} \in A$ and $\Upsilon(v_i) = \sum_l v_{il} \otimes f_{il}$.

Check that Δ_2 is well defined and that Δ_1 and Δ_2 are inverse isomorphisms of A comodules.

2.3

Theorem Let $(A, \mu_A, \varepsilon_A)$ and $(B, \mu_B, \varepsilon_B)$ be Hopf algebras over k and $\varphi : A \to B$ a Hopf algebra map such that $J = \ker \varphi$ is a normal Hopf ideal of A (see [13]). The identification $\varepsilon_B \otimes 1 : \varphi^0(B) \to A$ takes $\varphi^0(k)$ to a sub Hopf algebra of A.

Proof Put $V = (\varepsilon_B \otimes 1)\varphi^0(k) = \{a \in A \mid (\varphi \otimes 1)\mu_A(a) = 1 \otimes a\}$

$= \{a \in A \mid \mu_A(a) \in (J + k) \otimes A\}$.

Clearly V is a sub k algebra of A. Let $\{v_i\}_{i \in I}$ be a basis of V and let S be the antipode of A.

$\mu(v_i) = \sum_j v_j \otimes f_{ji}$, say. We have

$(\mu \otimes 1)\mu(v_i) = \sum_{j,l} v_l \otimes f_{lj} \otimes f_{ji}$, so J normal gives

$\sum_{j,l} v_l S(f_{ji}) \otimes f_{lj} \in A \otimes (J + k)$. Hence for any m

$$\sum_{j,l,i} v_l S(f_{ji}) f_{im} \otimes f_{lj} \in A \otimes (J + k) \text{ and so}$$

$$\sum_{j,l} v_l \delta_{jm} \otimes f_{lj} = \sum_l v_l \otimes f_{lm} \in A \otimes (J + k).$$

By this and a similar argument it follows that

$\{a \in A \mid \mu_A(a) \in (J + k) \otimes A\} = \{a \in A \mid \mu_A(a) \in A \otimes (J + k)\}$ (1).

From this it follows that V is a right and left subcomodule of A and it remains to show that V is S invariant.

Let $T : A \otimes A \to A \otimes A$ be the k map ddfined by

$$T(\sum_i a_i \otimes a_i') = \sum_i a_i' \otimes a_i \quad \text{for } a_i, a_i' \in A.$$

Since $\mu_A S = T(S \otimes S)\mu_A$ ([12, 4.0.1]) we have, for $a \in V$, $\mu_A(S(a)) \in T(S \otimes S)((J + k) \otimes A) = A \otimes (J + k)$. Hence by (1)

$S(a) \in V$.

<u>Corollary 1</u> If $(A,\mu,\mathcal{E})$ is a Hopf algebra over k, H a sub Hopf algebra and $J = A(H \cap \ker\mathcal{E})$ then if $\varphi: A \to A/J$ is the natural map and $\bar{\mathcal{E}}: A/J \to k$ is defined by $\bar{\mathcal{E}}(a + J) = \mathcal{E}(a)$ for any $a \in A$, we have $(\bar{\mathcal{E}} \otimes 1)\varphi^{o}(k) = H$.

<u>Proof</u> Put $H_1 = (\bar{\mathcal{E}} \otimes 1)\varphi^{o}(k)$. We may easily check that $H \leqslant H_1$ and $H_1 \cap \ker\mathcal{E} \leqslant J$. Hence $A(H_1 \cap \ker\mathcal{E}) = J$ and so by Takeuchi's theorem [13, 4.3], $H_1 = H$.

<u>Corollary 2</u> Let $(A,\mu,\mathcal{E})$ be a Hopf algebra over k, a perfect field of characteristic $p > 0$. Let $J = \mathfrak{M}^{[p^n]}$, $n \in \mathbb{N}$ and $\varphi : A \to A/J$ be the natural map. Then $(\bar{\mathcal{E}} \otimes 1)\varphi^{o}(k) = A^{(p^n)}$.

<u>Definition</u> Let $(A,\mu,\mathcal{E})$ be a Hopf algebra over K and let k be a subfield of K. A Hopf k form of A is a k subalgebra A_k having the following properties : 1) A_k contains a k basis which is also a K basis of A, 2) $\mu(A_k) \leqslant A_k \otimes A_k$, 3) $\mathcal{E}(A_k) \leqslant k$, 4) $S(A_k) = A_k$ where S is the antipode of A.

Say A is defined over k if A has a Hopf k form.

Suppose (A, μ, ε) is defined over k, the field of q elements. Let A_k be a Hopf k form, define $\varphi_k(a) = a^q$ for each $a \in A_k$. Let $\varphi: A \to A$ be the unique K linear extension of φ_k. One may easily check that φ is a Hopf algebra map.

<u>Proposition</u> Let (A, μ, ε) be as above and A a reduced ring. Let $\pi: A \to A/\mathfrak{m}[q]$ be the natural map. If X is an A comodule such that $\pi_o(X)$ is injective and Y an injective A comodule then $X \otimes \varphi_o(Y)$ is an injective A comodule.

<u>Proof</u> Y is injective and hence is a direct summand of a direct sum of copies of the left A comodule A. Thus by 2.2a it is enough to take $Y = A$.

One may check that $\varphi : \varphi_o(A) \to A^{(q)}$ is an isomorphism of A comodules. $A^{(q)} \cong \pi^o(K)$ by the second corollary to theorem 2.3. Now $X \otimes \pi^o(K) \cong \pi^o(\pi_o(X))$ by 2.2h, hence by 2.2f this is an injective A comodule.

<u>Remark</u> This proposition will be extremely useful in 3 where $A = K[G]$, the coordinate ring of a simply connected Chevalley group over K, an algebraically closed field of characteristic $p > 0$. We will be most interested in the case $p = q$, where φ_o corresponds to the Frobenius operation on rational G modules.

2.4 As an illustration of the restriction, induction machinery developed so far we give a proof of part of Steinberg's twisted tensor product theorem.

Our proof makes use of the following general result.

Lemma Let (R, μ, ε) be a Hopf algebra over K, algebraically closed, J a normal Hopf ideal, R_J the corresponding sub Hopf algebra (see [13]) and $\varphi: R \to R/J$ the natural map. If V is an R comodule such that $\varphi_o(V)$ is simple and W is a simple R comodule such that $cf(W) \leqslant R_J$ then $V \otimes W$ is simple.

Proof (Based on an idea of H. Blau).

Let $0 \neq U \leqslant V \otimes W$, U an R subcomodule of $V \otimes W$.

Since $cf(W) \leqslant R_J$, $\varphi_o(V \otimes W) \cong \varphi_o(V) \otimes (W)$ and so $\varphi_o(U)$ is a direct sum of copies of $\varphi_o(V)$.

Thus by the simplicity of $\varphi_o(V)$ and the algebraic closure of K, $\dim \mathrm{Hom}_{R/J}(\varphi_o(U), \varphi_o(V)) = \dim U / \dim V$ (1).

But $\mathrm{Hom}_{R/J}(\varphi_o(U), \varphi_o(V)) \cong \mathrm{Hom}_R(U, \varphi^o(\varphi_o(V)))$

$$\cong \mathrm{Hom}_R(U, V \otimes \varphi^o(K)) \cong \mathrm{Hom}_R(U, V \otimes R_J) \qquad (2)$$

by 2.2c, 2.2h and corollary 1 of the proposition of 2.3.

Now R_J may be regarded as a Hopf algebra in its own right, thus there is a decomposition

$$R_J = \sum_{\beta \in \underline{B},\ i=1,\dots \dim W_\beta} \oplus \ I_{\beta i} .$$

by [6, 1.5.1] , where $\{W_\beta\}_{\beta\in\underline{B}}$ is a full set of simple R_J comodules and $I_{\beta i} \cong I_\beta$ for $i = 1,\ldots \dim W_\beta$, $\beta \in \underline{B}$, and I_β is an R_J injective envelope of W_β . The $I_{\beta i}$ and W_β may be regarded as R comodules.

Now $W \cong W_\gamma$ for some $\gamma \in \underline{B}$ and so

$$\mathrm{Hom}_R(U,V\otimes R_J) \cong \sum_{i,\beta}{}^{\oplus} \mathrm{Hom}_R(U,V\otimes I_{\beta i}) \text{ implies}$$

$$\dim \mathrm{Hom}_R(U,V\otimes R_J) \geqslant (\dim W_\gamma)\dim \mathrm{Hom}_R(U,V\otimes W_\gamma).$$

Now $\mathrm{soc}_R\, I_\gamma \cong W_\gamma \cong W$ so

$$\dim \mathrm{Hom}_R(U,V\otimes R_J) \geqslant (\dim W) \dim \mathrm{Hom}_R(U,V\otimes W).$$

But $U \leqslant V\otimes W$ so by (1) and (2) we get

$\dim U / \dim V \geqslant \dim W$. However $U \leqslant V\otimes W$ gives $\dim U \leqslant (\dim V)(\dim W)$ thus $\dim U = (\dim V)(\dim W)$ and $U = V\otimes W$.

<u>Proposition</u> Let (A,μ,ε) be a Hopf algebra over K, algebraically closed of characteristic $p > 0$, suppose A is a reduced ring. Suppose A is defined over F_p, the field of p elements and let A_{F_p} be a Hopf F_p form. For $n > 0$ let $\varphi(n)$ be the unique K linear extension of the map $A_{F_p} \to A_{F_p}$ taking a to a^{p^n} for each $a \in A_{F_p}$. Let $\pi(n) : A \to A_n = A/\mathfrak{m}[p^n]$ be the natural map. Let V,W be A comodules such that $\pi(n)_o(V)$ is simple and $\pi(1)_o(W)$ is simple. Then $\pi(n+1)_o(V\otimes \varphi(n)_o(W))$ is a simple A_{n+1} comodule.

<u>Proof</u> $\pi(n+1)_o(V \otimes \varphi(n)_o(W))$

$\cong \pi(n+1)_o(V) \otimes (\pi(n+1)\varphi(n))_o(W)$ by 2.2g and 2.2(d)(i).

We now apply the above lemma with $R = A_{n+1}$ and $J = \mathfrak{m}^{[p^n]}/\mathfrak{m}^{[p^{n+1}]}$

Let $\pi(n,n+1) : A_{n+1} \to A_n$ be the map taking

$a + \mathfrak{m}^{[p^{n+1}]}$ to $a + \mathfrak{m}^{[p^n]}$ for any $a \in A$.

$\pi(n,n+1)_o(\pi(n+1)_o(V)) \cong (\pi(n,n+1)\pi(n+1))_o(V)$

by 2.2d(ii) $\cong \pi(n)_o(V)$ since $\pi(n,n+1)\pi(n+1) = \pi(n)$.

However $\pi(n)_o(V)$ is simple by hypothesis. It remains to show that $(\pi(n+1)\varphi(n))_o(W)$ is simple and that its coefficient space lies in $A_{n+1}{}^{(p^n)}$.

$cf((\pi(n+1)\varphi(n))_o(W)) \leq \pi(n+1)\varphi(n)(\,cf(W))$

$\leq \pi(n+1)\varphi(n)\ (A) = \pi(n+1)(\,A^{(p^n)}) = A_{n+1}{}^{(p^n)}$.

$\pi(n+1)\varphi(n) : A \to A_{n+1}$ has $\mathfrak{m}^{[p]}$ in its kernel, thus gives rise to a map $\xi : A_1 \to A_{n+1}{}^{(p^n)}$. By the argument of [11] second lemma of 2, it follows that ξ is an isomorphism of Hopf algebras. However

$(\pi(n+1)\varphi(n))_o(W) \cong (\xi\pi(1))_o(W)$

$\cong \xi_o\pi(1)_o(W)$, $\pi(1)_o(W)$ is simple by the hypotheses hence ξ is an isomorphism implies

$(\pi(n+1)\varphi(n))_o(W)$ is simple. This completes the proof of the proposition.

<u>Theorem</u> Let $(A,\mu,\varepsilon), \pi(n), \varphi(n)$ be as in the above proposition. Let $V_0, \ldots . V_n$ be A comodules such that each $\pi(1)_0(V_i)$ is simple, then

$$\pi(n+1)_0(V_0 \otimes \varphi(1)_0(V_1) \otimes \ldots \otimes \varphi(n)_0(V_n))$$

is a simple A_{n+1} comodule.

<u>Proof</u> By induction on n. For $n = 0$, $\pi(1)_0(V_0)$ is simple by hypothesis. Assume true for n-1. This gives

$$\pi(n)_0(V_0 \otimes \varphi(1)_0(V_1) \otimes \ldots \otimes \varphi(n-1)_0(V_{n-1})) \text{ is simple.}$$

Now in the above proposition we put

$$V = V_0 \otimes \varphi(1)_0(V_1) \otimes \ldots \ldots \otimes \varphi(n-1)_0(V_{n-1}), \quad W = V_{n-1}$$

to obtain that

$$\pi(n+1)_0(V_0 \otimes \varphi(1)_0(V_1) \otimes \ldots \varphi(n-1)_0(V_{n-1}) \otimes \varphi(n)_0(V_n))$$

is simple as required.

Let $A = K[G]$ for G a simply connected algebraic Chevalley group over K, algebraically closed of characteristic $p > 0$. Let $\{M(\lambda)\}_{\lambda \in X_p^+}$ be a full set of restricted simple rational modules for G, $M(\lambda)$ having unique highest weight $\lambda \in X_p^+$. Here X^+ is the set of dominant weights and X_p^+ the dominant weights which when expressed as a linear combination of fundamental dominant weights have all coefficients between 0 and p-1.

By theorem 6.4 of [3] $\pi(1)_0(M(\lambda))$ is simple for $\lambda \in X_p^+$ and so the above theorem gives

$\Pi(n+1)_o(M(\lambda_0) \otimes M(\lambda_1)^{Fr} \otimes \dots\dots\dots \otimes M(\lambda_n)^{Fr^n})$ is simple where Fr denotes the Frobenius operation on modules and each $\lambda_i \in X_p^+$. This, of course, is stronger than saying that $M(\lambda_0) \otimes M(\lambda_1)^{Fr} \otimes \dots \otimes M(\lambda_n)^{Fr^n}$ is simple as an R comodule which is part of the Steinberg twisted tensor product theorem.

3 F Complements and Chevalley Groups

3.1

Let (R,μ,ε) be a coalgebra over k, a field. Let (V,τ) be a finite dimensional right (resp. left) R comodule. $V^* = \mathrm{Hom}_k(V,k)$ may be given a right (resp. left) R comodule structure (see [6, 1.2]). As in [6, 1.2] we define a k linear map $c : V^* \otimes V \to cf(V)$ by

$$c(\alpha \otimes v) = (\alpha \otimes 1)\tau(v), \text{ where } \alpha \in V^* \text{ and } v \in V.$$

Note c is onto and if we regard $V^* \otimes V$ as the left comodule $(V^*) \otimes V$ and as the right comodule $V^* \otimes (V)$ then c is an R^* bimodule map. We denote $c(\alpha \otimes v)$ by $c_{\alpha,v}$ $(\alpha \in V^*, v \in V)$ and observe $a \circ c_{\alpha,v} \circ b = c_{\alpha \circ b, a \circ v}$ for $a,b \in R^*$. We may easily show that if $c_{\alpha,v} = 0$ for all $\alpha \in V^*$ then $v = 0$.

We now put $R = K[G]$, the coordinate ring of a simply connected Chevalley group G over K, an algebraically closed field of characteristic $p > 0$. Put $R_n = R/\mathfrak{m}^{[p^n]}$ for $n \in \mathbb{N}$. We identify R_n^* with the subspace of R^* consisting of those elements which vanish on $\mathfrak{m}^{[p^n]}$ and further with the $\underset{\sim}{u}_n$ of [9]. The identification $\varphi : \underset{\sim}{u}_n \to R_n^*$ is given on the generators $X_{\alpha,i}$ (see [9], $0 \leq i \leq p^n-1$, α a root by

$$\sum_{i=0}^{\infty} t^i \varphi(X_{\alpha,i})(r) = r(x_\alpha(t)) \text{ for } r \in R \text{ and}$$

elements $x_\alpha(t)$ of the root subgroup X_α.

Roots and root subgroups are computed with respect to the standard maximal torus of diagonal matrices H arising from the Chevalley construction, U (resp. U') denotes the maximal unipotent subgroup which consists of upper (resp. lower) triangular matrices arising from the Chevalley construction. We identify G with a subset of R^* by $g \to \varepsilon_g$ where $\varepsilon_g(r) = r(g)$ for $g \in G$, $r \in R$.

<u>Definition</u> $N_0^+(n)$ (resp. $N_0^-(n)$) = the K span of products of the elements $X_{\alpha,i}$ for $0 \leq i \leq p^n-1, \alpha > 0$ (resp. $\alpha < 0$).

$N^+(n) = N_0^+(n) + K\varepsilon$, $N^-(n) = N_0^-(n) + K\varepsilon$. When n is clear we will omit it from the notation just given.

<u>Lemma 1</u> (J.A.Green) Let L,M be left R_n comodules such that (i) $L = \underset{\sim}{u}_n \circ 1$ where $N_0^+ \circ 1 = 0$

and (ii) $M = N^+ \circ m$ where $N_0^- \circ m = 0$.

Then $L \otimes M = \underset{\sim}{u}_n \circ (1 \otimes m)$.

<u>Proof</u> We first find an expression for the action of $X_{\alpha,i}$ on a tensor product of R_n comodules. Let (u_n, Δ, e) be the noncommutative Hopf algebra dual to R_n. If V,W are R_n comodules and $\gamma \in \underset{\sim}{u}_n$ then $\gamma \circ (v \otimes w) = \sum_i \gamma_i \circ v \otimes \gamma_i' \circ w$ for $v \in V$, $w \in W$ and $\Delta(\gamma) = \sum_i \gamma_i \otimes \gamma_i'$.

Now for $r,s \in R$ $(rs)(x_\alpha(t)) = r(x_\alpha(t))s(x_\alpha(t))$ $= (\sum_u t^u X_{\alpha,u}(r))(\sum_v t^v X_{\alpha,v}(s))$. Coefficient of t^i is

$\sum_j X_{\alpha,j}(r)\, X_{\alpha,i-j}(s)$. Hence $\Delta(X_{\alpha,i}) = \sum_{j\leq i} X_{\alpha,j} \times X_{\alpha,i-j}$

Put $M_1 = \{y \in M \mid 1 \otimes y \in Z\}$ where $Z = u_n \circ (1 \otimes m)$.
If $y \in M_1$, $1 \otimes y \in Z$, so for $\alpha > 0$, $X_{\alpha,i} \circ (1 \otimes m)$
$= 1 \otimes X_{\alpha,i} \circ y \in Z$. Hence $X_{\alpha,i} \circ y \in M_1$ and so $m \in M_1$ gives
$N^{+} \circ m = M \leq M_1$ by (ii) and so $M_1 = M$. Put

$L_1 = \{x \in L \mid x \otimes M \leq Z\}$, by a similar argument we get $L_1 = L$
so $Z = L \otimes M$.

<u>Remark</u> Clearly the right handed version of this lemma works just as well.

<u>Definition</u> If X is a left comodule for R_n we define X^+ (resp. X^-) to be $\{r \in cf(X) \mid r \circ N_0^- = 0$ (resp. $r \circ N_0^+ = 0)\}$.

Note if X is simple $cf(X)$ is a direct sum of copies of X by [6, 1.3a(ii)]. Every simple u_n module is the restriction of a simple R comodule ([9,2]) and it follows from the structure of such comodules that $X \cong X^+ \cong X^-$, as left R_n comodules.

$\{\xi \in X^* \mid \xi \circ X_{\alpha,i} = 0$, all $\alpha < 0$, $0 \leq i \leq p^n-1\} = Kl$ for some $l \neq 0$. Further $c : X^* \otimes X \to cf(X)$ is an isomorphism ([6,1.3.1]) and so $X^+ = \{c_{l,x} \mid x \in X\}$. Similarly for Y a simple left R_n comodule we have $Y^- = \{c'_{m,y} \mid y \in Y\}$ where $0 \neq m \in Y^*$ is such that $m \circ X_{\alpha,j} = 0$ for all $\alpha > 0$, $0 \leq j \leq p^n-1$ and $c' : Y^* \otimes Y \to cf(Y)$ is the natural map.

Lemma 2 (J.A.Green) Let X, Y be simple R_n comodules as above,

define $\psi : X \otimes Y \to X^+ \otimes Y^- \to X^+Y^-$

$$x \otimes y \to c_{1,x} \otimes c'_{m,y} \to c_{1,x}c'_{m,y} .$$

ψ is an isomorphism of R_n comodules.

Proof $cf(X \otimes Y) = cf(X)cf(Y)$ so we have a map

$d : (X \otimes Y)^* \otimes (X \otimes Y) \to cf(X)cf(Y)$ and we note that

$d_{\xi \otimes \eta, x \otimes y} = c_{\xi, x}c'_{\eta, y}$ for $\xi \in X^*$, $\eta \in Y^*$, $x \in X$,

$y \in Y$. So if $z \in X \otimes Y$, $\psi(z) = d_{1 \otimes m, z}$.

Now if $\psi(z) = 0$, $d_{1 \otimes m, z} \circ b = 0$ for all $b \in u_n$. Hence

$d_{(1 \otimes m) \circ b, z} = 0$ for all $b \in u_n$. By the right handed version of

lemma 1 we have $d_{\gamma, z} = 0$ for all $\gamma \in (X \otimes Y)^*$. So by a

previous remark $z = 0$. Thus ψ is an injective map. By

dimensions it is an isomorphism.

Definition For V a left $R = K[G]$ comodule we define

V^+ (resp. V^-) to be $\{r \in cf(V) \mid r \circ g = r$ for all $g \in U'\}$ (resp. U)

Definition $R(0) = \{r \in R \mid r \circ h = r$ for all $h \in H\}$. $R(0)$

may be identified with $K[G/H]$ and with $\varphi^0(K)$ where

$\varphi : R \to K[H]$ is the restriction map.

Theorem $R(0)$ is a U.L. augmentation algebra for which

St^+St^- is an F complement.

Remarks St_n denotes the simple rational G module (i.e. R

comodule) having unique highest weight $(p^n-1)\delta$ where δ is

the half sum of the positive roots. We write simply St for St_1.

$R(0)$ is an augmentation algebra on which G acts as a group of K algebra automorphisms. St^+St^- is a G submodule of $R(0)$. In view of this structure we refer to St^+St^- as an F complement rather than simply a complement.

<u>Proof of theorem</u> Let $\rho_n : R \to R_n$ be the natural map. We first show $1 \in St_n^+St_n^-$. By arguments similar to those of the above lemmas $St_n^+St_n^- \cong St_n \otimes St_n$. Thus since St_n is self dual we have $\mathrm{Hom}_G(K, St_n \otimes St_n) \neq 0$ and so $K.1 \in St_n^+St_n^-$. Now $\rho_n(1) \neq 0$ gives $\rho_n(St_n^+) \neq 0$. St_n is simple as an R_n comodule (or $R_n^* = u_n$ module) by [9, Prop 2.3] and $\rho_n : R \to R_n$ is an R_n comodule map. It follows that $\rho_n(St_n^+) = [(\rho_n)_o(St_n)]^+$. Hence by the preceding lemma $\rho_n(St_n^+St_n^-) = ((\rho_n)_o(St_n))^+((\rho_n)_o(St_n))^-$ has dimension $(\dim St_n)^2 = p^{2n \dim U}$. Since ρ_n restricted to $St_n^+St_n^-$ is injective we have $St_n^+St_n^- \cap \mathfrak{M}^{[p^n]} = 0$, where $\mathfrak{M}$ is the augmentation ideal of R. Hence

$$(1) \quad \dim R(0)/(R(0) \cap \mathfrak{M})^{[p^n]} \geq p^{n\, 2\dim U}.$$

But $R(0) = K[G/H]$ so since $\dim G/H = 2 \dim U$ we get $\dim R(0) \cap \mathfrak{M} / (R(0) \cap \mathfrak{M})^2 = 2 \dim U$ (the dimension of the tangent space at 1 = the dimension of G/H since G/H is a smooth affine variety).

Now by remark 1 of 1.1 we have the reverse inequality to (1). Hence (1) is an equality and $R(0)$ is a U.L.

augmentation algebra. (augmentation map is the restriction of $\mathcal{E}$). Now by dimensions and the fact that $St^+St^- \cap (R(0) \cap \mathfrak{m})^{[p]} = 0$ we have St^+St^- is an F complement of R(0).

Lemma 3 Let W be a finite dimensional rational G module (i.e. an R comodule). Let $W = \sum_{\mu}^{\oplus} W_\mu$ be a weight space decomposition of W, W_μ being the space of vectors of weight μ. For all n sufficiently large

$$\dim \operatorname{Hom}_G(St_n, W \otimes St_n) = \dim W_0.$$

Proof Put $m(\mu) = \dim W_\mu$ for a weight μ and let X(H) denote the lattice of all weights. Pick n large enough so that

(a) $\mu + (p^n-1)\delta$ is dominant for each weight μ of W and

(b) $m(\mu) \neq 0$ and $\mu \in p^n X(H)$ implies $\mu = 0$.

Let $\underline{g}_C$ be the complex Lie algebra from which G has been obtained via the Chevalley construction. For a dominant weight let $V(\lambda)$ be a finite dimensional simple $\underline{g}_C$ module of highest weight λ. Let $V(\overline{\lambda})$ be a reduction mod p of $V(\lambda)$, reduced via a minimal admissible lattice. For a rational finite dimensional G module M, ch(M) denotes the character of M (computed relative to H).

It follows from the proposition of 7.2 of [7] that

(1) $$ch(W)ch(St_n) = \sum_{\mu} m(\mu) ch(\mu + (p^n-1)\delta)$$

where $ch(\nu) = ch\, V(\overline{\nu})$ for a dominant weight ν.

Now as a $\underset{\sim}{u}_n$ module St_n is projective and simple. It is therefore the unique simple $\underset{\sim}{u}_n$ module in its block, say $B(e)$ where e is the corresponding central idempotent of $\underset{\sim}{u}_n$.

G acts rationally on u_n by $\theta^g = \varepsilon_{g^{-1}}\,\theta\,\varepsilon_g$ for $\theta \in \underset{\sim}{u}_n$ and $g \in G$. It acts as a group of K algebra automorphisms and hence permutes the central idempotents. Since G is connected and $\underset{\sim}{u}_n$ has only finitely many central idempotents G fixes each central idempotent. It follows that any $\underset{\sim}{u}_n$ block component of a rational G module is a G component.

Let Y be the B(e) block component of $W \otimes St_n$. From (1) we deduce that St_n occurs at least $m(0)$ times as a G composition factor of $W \otimes St_n$.

Hence (2) $\dim Y / \dim St_n \geqslant m(0)$.

Now suppose M is a G composition factor of Y. Since Y as a $\underset{\sim}{u}_n$ module is a direct sum of copies of St_n we must have

(3) $M \cong St_n \otimes M(\lambda)^{Fr^n}$, for some dominant weight λ where Fr denotes the Frobenius operator on modules. Here $M(\lambda)$ denotes the simple rational G module having highest weight λ. (3) follows from the Steinberg twisted tensor product theorem and the proposition in 2.3 of [9].

Since $V(\overline{\lambda'})$ is an indecomposable G module for any dominant λ', a G composition factor of $V(\overline{\lambda'})$ is in B(e) if and only if $V(\overline{\lambda'})$ is in B(e) if and only if $M(\overline{\lambda'})$ is in B(e).

If $M((p^n-1)\delta+\mu)$ is in $B(e)$ for a weight μ of W then $(p^n-1)\delta+\mu = (p^n-1) \quad + p^n\lambda'$ for some dominant weight λ' by (3). Hence $\mu \in p^nX(H)$ and so $\mu = 0$ by (b). Now (1) gives (4) $\dim Y / \dim St_n = m(0)$. From (3) and (4) we obtain that all G composition factors of Y are isomorphic to St_n. Thus by the lemma of 4.1 of [9] Y is completely reducible and it follows that

$$\dim \mathrm{Hom}_G(St_n, W \otimes St_n) = m(0).$$

<u>Remark</u> The argument used in the above lemma is similar to, and modelled on, the argument used by Humphreys in 5.1 of [9].

<u>Proposition</u> The F complement St^+St^- for $R(0)$ is strong.

<u>Proof</u> Put $B = (St^+St^-)_\infty$. Let V be any finite dimensional rational G module, then

$\mathrm{Hom}_G(V,(St^+St^-)_n) \cong \mathrm{Hom}_G(V,St_n \otimes St_n) \cong \mathrm{Hom}_G(St_n{}^*,V^* \otimes St_n)$
$\cong \mathrm{Hom}_G(St_n, V^* \otimes St_n)$ (1) since St_n is self dual.
For a finite dimensional rational left G module W we are using W^* to denote the left dual module.

By (1) and lemma (3) we have

$$\dim \mathrm{Hom}_G(V, (St^+St^-)_n) = m(0) \text{ for all } n \text{ large enough,}$$

where $m(0)$ is the dimension of the zero weight space of V.

Hence $\dim \mathrm{Hom}_G(V,B) = m(0)$.

Note also that if $\varphi : R \to K[H]$ is the restriction map

$$\dim \mathrm{Hom}_G(V, R(0)) = \dim \mathrm{Hom}_G(V, \varphi^0(K))$$
$$= \dim \mathrm{Hom}_H(\varphi_0(V), K) = m(0).$$

Hence the natural map $\mathrm{Hom}_G(V,B) \to \mathrm{Hom}_G(V,R(0))$ is an identification. Now suppose that $B \neq R(0)$. Pick $v \in R(0)\setminus B$ and put $V = KG.v$. Consider the embedding $i : V \to R(0)$ given by $i(v) = v$ for all $v \in V$. Since $V \not\subseteq B$

$i \in \mathrm{Hom}_G(V,R(0))\setminus \mathrm{Hom}_G(V,B)$ which is a contradiction.

Thus $B = R(0)$ and St^+St^- is a strong F complement.

3.2

Let $X(H)_p$ denote the set of restricted dominant weights. For $\mu \in X(H)_p$ we define $\hat{\mu}$ to be $(p-1)\delta + w_0(\mu)$ where w_0 is the unique element of the Weyl group of G which takes all positive roots to negative roots.

$M(\mu)$ occurs exactly once as a G and u_1 submodule of $M(\hat{\mu}) \otimes St$ [7, 8.2]. Let $M(\hat{\mu}) \otimes St = Y(\mu) \oplus Z$ be a G decomposition such that $Y(\mu)$ is indecomposable and $M(\mu)$ occurs as a G submodule of $Y(\mu)$. We call $Y(\mu)$ the infinitesimally injective cover of $M(\mu)$. It is determined up to G isomorphism by the Krull-Schmidt theorem.

Theorem [1] Let h be the Coxeter number of G. If $p > 2h - 2$ then, for each $\mu \in X(H)_p$, $Y(\mu)$ regarded as a u_1 module is the u_1 injective cover of $M(\mu)$.

We will say that hypothesis (H) holds if, for all $\mu \in X(H)_p$ $Y(\mu)$ regarded as a $\underset{\sim}{u}_1$ module is the $\underset{\sim}{u}_1$ injective cover of $M(\mu)$.

<u>Note</u> (H) is true in all known cases and conjectured to be true in general. We assume for the rest of the section that G is a group for which (H) is true.

<u>Theorem</u> Let $Y(0)$ be realised inside St^+St^-. $I = Y(0)_\infty$ is the rational G injective cover of $M(0)$. The rational G injective cover of $M(\lambda_0) \otimes M(\lambda_1)^{Fr} \otimes \ldots \otimes M(\lambda_n)^{Fr^n}$, where all $\lambda_i \in X(H)_p$ and $\lambda_n \neq 0$ is

$$Y(\lambda_0) \otimes Y(\lambda_1)^{Fr} \otimes \ldots\ldots \otimes Y(\lambda_n)^{Fr^n} \otimes I^{Fr^{n+1}}.$$

<u>Proof</u> $Y(0) \mid St^+St^-$ hence by proposition (b) of 1.2 $I = Y(0)_\infty \mid (St^+St^-)_\infty$. By the proposition of 3.1 $(St^+St^-)_\infty = R(0)$ which is injective (since it is a G summand of R).

Now by repeated application of the proposition of 2.3 we get $Y(\lambda_0) \otimes Y(\lambda_1)^{Fr} \otimes \ldots\ldots \otimes Y(\lambda_n)^{Fr^n} \otimes I^{Fr^{n+1}}$ is injective. Put $\lambda = \sum_{i=0}^{n} p^i\lambda_i$ and

$$I(\lambda) = Y(\lambda_0) \otimes Y(\lambda_1)^{Fr} \otimes \ldots \otimes Y(\lambda_n)^{Fr^n} \otimes I^{Fr^{n+1}}.$$

It remains to prove that the G socles of I and $I(\lambda)$ are simple.

By the argument of proposition (a) of 1.2 $Y(0)_m \cong Y(0) \otimes Y(0)^{Fr} \otimes \ldots \otimes Y(0)^{Fr^{m-1}}$. Hence by the theorem of [8, 1.1] $\mathrm{soc}_{\underset{\sim}{u}_m} Y(0)_m \cong M(0)$ as $\underset{\sim}{u}_m$ modules for any $m > 0$.

Now by a Clifford's theorem argument

$soc_G(Y(0)_m) \leqslant soc_{\underset{\sim}{u}_m}(Y(0)_m)$ and so $soc_G(Y(0)_m) \cong M(0)$.

Finally since $I = \bigcup_{m>0} Y(0)_m$, $soc_G(I) \cong M(0)$.

A similar argument shows that

$$soc_G(I(\lambda)) \cong M(\lambda_0) \otimes M(\lambda_1)^{Fr} \otimes \ldots \otimes M(\lambda_n)^{Fr^n}.$$

<u>Notes</u> 1. We did not use the full strength of hypothesis (H). in proving the theorem. It is enough to assume that for each $\mu \in X(H)_p$, $Y(\mu)$ is a G module which when viewed as a $\underset{\sim}{u}_1$ module is isomorphic to the u_1 injective cover of $M(\mu)$ and that $Y(0)$ is a G summand of $St \otimes St$.

2. This theorem was first proved for $SL_2(K)$ by P.W.Winter [14]. It was conjectured to be true generally by Humphreys. A version of it has been proved by J.W.Ballard [1].

<u>3.3</u>

We now apply the proposition of 3.3 to produce a strong F complement for $K[U]$.

<u>Lemma</u> Let $\pi : R \to K[U]$ be the restriction map.

$\pi(R(0)) = K[U]$.

<u>Proof</u> (a) $G = SL_n(K)$.

U = the unipotent upper triangular matrices in $SL_n(K)$

H = the diagonal matrices in $SL_n(K)$.

Let X_{ij} be the coordinate functions for G, $1 \leqslant i,j \leqslant n$.

Let Y_{ij}, $1 \leq i < j \leq n$ be the coordinate functions for U.

It is easy to see that

$$Z_{ij} = X_{11} \cdots X_{i-1,i-1} X_{ij} X_{i+1,i+1} \cdots X_{nn} \in R(0).$$

Moreover for $i < j$, $\pi(Z_{ij}) = Y_{ij}$. $K[U]$ is generated by the Y_{ij} and π is a K algebra map, hence $\pi(R(0)) = K[U]$.

(b) Let G be an arbitrary simply connected Chevalley group over K. $G \leq SL_n(K)$ and

$U \leq U_0$ = the upper triangular unipotent matrices

$H \leq H_0$ = the diagonal matrices in $SL_n(K)$.

Put $G_0 = SL_n(K)$, $K[G_0]_{(H_0)}$ = the elements of $K[G_0]$ fixed by the right H_0 action. The diagram

$$\begin{tikzcd} K[G_0]_{(H_0)} \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\bullet"] & K[U_0] \arrow[d, "\beta"] \\ R(0) \arrow[r, "\gamma"'] & K[U] \end{tikzcd}$$

commutes where all maps are restrictions. Moreover α and β are onto, hence γ is onto and the result is established.

<u>Theorem</u> $\pi(St^+)$ is a strong F complement for $K[U]$.

<u>Proof</u> First observe that if $f \in St^-$ then for $u \in U$,

$f(u) = f \circ u(1) = f(1)$ so $\pi(St^-) \leq K$.

Now $1 \in St^+St^-$ and $\pi(1) = 1$ so $\pi(St^+St^-) \neq 0$. It follows that $\pi(St^-) = K$ and $\pi(St^+) = \pi(St^+St^-)$.

Let $\mathfrak{M}_0$ = the augmentation ideal of $R(0)$

$\mathfrak{M}_U$ = the augmentation ideal of $K[U]$.

$St^+St^- + \mathfrak{M}_0^{[p]} = R(0)$ (theorem of 3.1) and so

$\pi(St^+St^-) + (\mathfrak{M}_0^{[p]}) = \pi(R(0)) = K[U]$ by the above lemma.

But $\pi(\mathfrak{M}_0^{[p]}) \subseteq \mathfrak{M}_U^{[p]}$ hence

(1) $\pi(St^+St^-) + \mathfrak{M}_U^{[p]} = K[U]$.

Now $\dim \pi(St^+St^-) = \dim \pi(St^+) \leqslant p^{\dim U}$ and

$\dim K[U] / \mathfrak{M}_U^{[p]} = p^{\dim U}$.

Hence the sum of (1) is direct and $\pi(St^+) = \pi(St^+St^-)$ is an F complement. It is strong since

$$(\pi(St^+St^-))_\infty = \pi((St^+St^-)_\infty) = \pi(R(0)) = K[U].$$

3.4

Let V be an F complement for a U.L. Hopf algebra (A,μ,ι) over K.
Let $\rho : A \to A_1 = A/\mathfrak{M}^{[p]}$ be the natural map.

Definition Say V is a full F complement for A if there is an A comodule decomposition

$$V = V(0) \oplus V(1) \oplus \ldots\ldots \oplus V(n)$$

such that each $\rho_0(V(r))$ is indecomposable (as an A_1 comodule).

Theorem $R = K[G]$ has a strong and full F complement.

Proof Let $R = \sum_{\lambda \in X(H)^+}^{\oplus} (I(\lambda,1) \oplus \ldots\ldots \oplus I(\lambda,m_\lambda))$

be a decomposition of R into injective indecomposables, where

$soc_R(I(\lambda,i)) \cong M(\lambda)$, $m_\lambda = \dim M(\lambda)$ and $X(H)^+$ denotes the set of dominant weights. For $\lambda \in X(H)_p^*$, $1 \leqslant i \leqslant m$ let $Y(\lambda,i)$ be a copy of the infinitesimally injective comodule $Y(\lambda)$ realised inside $I(\lambda,i)$. The existence of such a $Y(\lambda,i)$ is ensured by the injectivity of $I(\lambda,i)$ and the fact that $soc_R(Y(\lambda)) \cong soc_R(I(\lambda,i))$.

Put $$V = \sum_{\lambda\in X(H)_p} \oplus (Y(\lambda,1) \oplus \ldots\ldots \oplus Y(\lambda,m_\lambda)) \quad (1).$$

Claim V is a strong and full F complement for R.

V contains 1 since it contains a copy of $M(0)$.

$$\mathrm{Dim}\ V = \sum_{\lambda\in X(H)_p} \dim M(\lambda) \dim Y(\lambda) = \dim \mathfrak{u}_1 = p^{\dim G}.$$

This is true since $\{M(\lambda)\}_{\lambda\in X(H)_p}$ is a full set of simple $R/\mathfrak{m}[p]$ comodules [3, 6.6] and $Y(\lambda)$ as an $R/\mathfrak{m}[p]$ comodule is an $R/\mathfrak{m}[p]$ injective cover of $M(\lambda)$ for each $\lambda \in X(H)_p$.

To complete the proof that V is an F complement we must show that $V \cap \mathfrak{m}^{[p]} = 0$.

Now V and $\mathfrak{m}^{[p]}$ are $\mathfrak{u}_1$ modules so it is enough to prove that $soc_{\mathfrak{u}_1}(V) \cap \mathfrak{m}^{[p]} = 0$.

$$soc_{\mathfrak{u}_1}(V) = \sum_{\lambda,i} \oplus\, soc_{\mathfrak{u}_1}(Y(\lambda,i)) = \sum_{\lambda\in X(H)_p} \oplus\ cf(M(\lambda)).$$

Since $\{M(\lambda)\}_{\lambda\in X(H)_p}$ remain simple on restriction to $\mathfrak{u}_1$ it follows that the natural map $\sum_{\lambda\in X(H)_p} \oplus\ cf(M(\lambda)) \to R/\mathfrak{m}[p]$ is an injection. Hence $soc_{\mathfrak{u}_1}(V) \cap \mathfrak{m}^{[p]} = 0$.

Thus V is an F complement moreover it is full by construction. Using the decomposition (1) and proposition (b) of 1.2 we may decompose V_∞ as the direct sum of comodules, each of which is injective by the theorem of 3.2. Hence V_∞ is injective.

Put $W = \mathrm{soc}_R(V) = \sum_{\lambda \in X(H)_p} \oplus\, \mathrm{cf}(M(\lambda))$.

We may deduce from Steinberg's twisted tensor product theorem that $\mathrm{soc}_R(R) = W_\infty$. Hence $\mathrm{soc}_R(R) \leq V_\infty$. Now it follows that an R comodule complement to V_∞ in R must have zero socle, hence $V_\infty = R$.

Remarks 1 We have shown that the decomposition (1) gives rise to, via proposition (b) of 1.2, a decomposition of R into injective indecomposables.

2 This method of producing an F complement is, in practice, non constructive. We would very much like to be able to write down, in terms of polynomials in the coordinate functions, F complements for simply connected Chevalley groups other than SL_2.

4. Remarks on Blocks

As usual G is a simply connected algebraic Chevalley group over K, an algebraically closed field of chacteristic $p > 0$. Let $\{I(\lambda)\}_{\lambda \in X(H)^+}$ be a full set of injective indecomposable $K[G]$ comodules such that $\mathrm{soc}_G(I(\lambda)) \cong M(\lambda)$ for each $\lambda \in X(H)^+$.

For $\lambda, \lambda' \in X(H)^+$, $c_{\lambda'\lambda} = \dim \mathrm{Hom}_G(I(\lambda), I(\lambda'))$ (see [6,2.5]). It follows from [6,2.5.5] that $c_{\lambda'\lambda} \neq 0$ if and only if $c_{\lambda\lambda'} \neq 0$ (in fact as Winter suggests [15,p63] $c_{\lambda\lambda'}$ is either 0 or ∞). We say λ is adjacent to λ' if $c_{\lambda\lambda'} \neq 0$ and let $\sim$ be the equivalence relation generated by adjacency. We say $M(\lambda')$ and $M(\lambda'')$ are in the same block or λ' and λ'' are in the same block if $\lambda' \sim \lambda''$. We think of a block as an equivalence class of $X(H)^+$ under $\sim$.

Let B be a block, we define

$$B^{\theta} = \{(p-1)\delta + p\lambda \mid \lambda \in B\}.$$

<u>Proposition</u> If B is a block then B^{θ} is a block.

<u>Proof</u> To prove this we use the following fact (*) which will be proved later.

(*) If $\lambda \in X(H)^+$ then $I((p-1)\delta + p\lambda) \cong St \otimes I(\lambda)^{Fr}$.

Suppose that λ' and λ'' are adjacent, i.e. $M(\lambda'')$ is a composition factor of $I(\lambda')$. Using (*) and the Steinberg twisted tensor product theorem we see that $M((p-1)\delta + p\lambda'')$ is a composition factor of $I((p-1)\delta + p\lambda') \cong St \otimes I(\lambda)^{Fr}$. Hence $(p-1)\delta + p\lambda''$ and $(p-1)\delta + p\lambda'$ are adjacent and so since $\sim$ is the equivalence relation generated by adjacency any two elements of B^{θ} are equivalent.

We must show that if $\lambda' \in B^{\theta}$ and $\lambda'' \sim \lambda'$ then $\lambda'' \in B^{\theta}$. Clearly we may assume that λ'' is adjacent to λ'.

$\lambda' \in B^{\theta}$ implies that $\lambda' = (p-1)\delta + p\lambda$ for some $\lambda \in B$ and so $I(\lambda') \cong St \otimes I(\lambda)^{Fr}$. But now $M(\lambda'')$ is a composition factor of $I(\lambda')$ implies that $\lambda'' = (p-1)\delta + p\mu$ for some μ such that $M(\mu)$ is a composition factor of $I(\lambda)$, by Steinberg's twisted tensor product theorem. Thus $\lambda'' \in B^{\theta}$.

Definition (see[7,3.1,3.2]) If $\lambda, \lambda' \in X(H)$ we say that λ and λ' are linked if there is a $w \in W$, the Weyl group, such that $w(\lambda + \delta) = \lambda' + \delta \mod pX(H)$. Clearly linkage defines an equivalence relation.

We have the following linkage principle ([7,3.1,3.2]) ; if $M(\lambda)$ and $M(\mu)$ are composition factors of an indecomposable rational G module then λ and μ are linked in X.

Let L be any linkage class of $X(H)$ not containing $(p-1)\delta$, put $B_{i,L} = \{\lambda \in X(H)^{+} \mid \lambda = \lambda_0 + p\lambda_1 + \dots + p^{i-1}\lambda_{i-1} + p^{i}\lambda_i + \dots + p^{r}\lambda_r$ where $\lambda_0 = \lambda_1 = \dots = \lambda_{i-1} = (p-1)\delta$, $\lambda_i \in L$ and all $\lambda_j \in X(H)_p\}$.

The following theorem has been proved by Humphreys in [8] using the theorem of 3.2; we now give a proof which does not involve theorem 3.2 and hence proves the theorem for all primes.

<u>Theorem</u> Weights belonging to a given block of $K[G]$ all lie in one of the sets $B_{i,L}$.

<u>Proof</u> By induction on i. For $i = 0$ this is just the linkage principle. Suppose true for $i \geqslant 0$, then

$$B_{i,L} = \dot{\bigcup_{\alpha \in J_{i,L}}} B_\alpha \quad \text{say where } B_\alpha \text{ are the blocks of } K[G]$$

lying in $B_{i,L}$. Now apply the operation θ to both sides to get
$$B_{i+1,L} = \dot{\bigcup_{\alpha \in J_{i,L}}} B_\alpha^\theta \quad .$$
Thus B_{i+1} is a union of blocks by the above proposition and we are done.

<u>Notes</u> 1 Humphreys suggests that after further partitioning the $B_{i,L}$ corresponding to cosets of the root lattice in $X(H)$ we should obtain precisely the blocks of $K[G]$.

2 By the proposition of 2.3 $St \otimes I(\lambda)^{Fr}$ is injective for any $\lambda \in X(H)^+$. We may show it is isomorphic to $I((p-1)\delta + p\lambda)$ either by showing it has a simple socle or that it is indecomposable. We may show it has a simple socle by a generalisation of the lemma of 2.4. However for the sake of variety we adopt the second approach and proceed by the following general result.

<u>Theorem</u> Let G be an algebraic group over K, an algebraically closed field, $R = K[G]$, J a normal Hopf ideal, R_J the corresponding sub Hopf algebra and $\varphi : R \to R/J$ the natural map. Suppose V and W are R comodules such that $\varphi_o(V)$ is an indecomposable R/J comodule of finite width and W is an indecomposable R comodule of finite width such that $\mathrm{cf}(W) \leq R_J$. Then $V \otimes W$ is an indecomposable R comodule of finite width.

<u>Proof</u> We first show that $V \otimes W$ has finite width (see (B) definition in 1.1). Let M be a finite dimensional R comodule. Since $\varphi_o(V)$ has finite width there is a finite dimensional subspace $X \leq \varphi_o(V)$ such that

$$\Theta(M) \subseteq X \text{ for all } \Theta \in \mathrm{Hom}_{R/J}(\varphi_o(M), \varphi_o(V)).$$

It follows that if V_1 is a finite dimensional subcomodule of V containing X $\mathrm{Hom}_R(M, V \otimes W)$ may be identified with $\mathrm{Hom}_R(M, V_1 \otimes W)$. But

$\mathrm{Hom}_R(M, V_1 \otimes W) \cong \mathrm{Hom}_R(M \otimes V_1^*, W)$ which is finite dimensional by the proposition of 1.1 of (B), us ing the fact that W is an R comodule of finite width. Hence for any finite dimensional R comodule M, $\mathrm{Hom}_R(M, V \otimes W)$ is finite dimensional and so $V \otimes W$ has finite width.

Put $E = \mathrm{End}_R(V \otimes W)$, $D = \mathrm{End}_{R/J}(\varphi_o(V))$ and $F = \mathrm{End}_R(W)$.

G acts on D by conjugation, i.e. for $g \in G$, $\alpha \in D$, α^g is defined by $\alpha^g(v) = g^{-1}\alpha(gv)$ for $v \in V$. Now $\varphi_0(V)$ is indecomposable of finite width and so by (B) 1.3 proposition 1 $D/J(D) \cong K$, where $J(D)$ is the Jacobson radical of D.

Let $\lambda : D \to K$ be the natural map. $J(D)$ is fixed under conjugation by g since g acts as an algebra automorphism of D, thus $\lambda(\alpha^g) = \lambda(\alpha)$ (1) for any $\alpha \in D$, $g \in G$.

Let $\{x_i\}_{i\in I}$ be a fixed basis of W, then if $\theta \in E$ we define K endomorphisms θ_{ij} for $(i,j) \in I \times I$ by

$$\Theta(v \otimes x_i) = \sum_{j\in I} \theta_{ji}(v) \otimes x_j.$$

Now $cf(W) \leq R_J$ implies $\varphi_0(V \otimes W) \cong \varphi_0(V) \otimes (W)$ and so each θ_{ij} is an R/J map.

Define $\Lambda : E \to F$ by

$$\Lambda(\Theta)(x_i) = \sum_{j\in I} \lambda(\theta_{ji})x_j. \quad \text{for } \theta \in E.$$

There are two things to check : (a) that the sum is finite; and (b) that $\Lambda(\Theta)$ is an R map. (a) follows from an argument similar to one used in the theorem of 3.1 of (B). We now prove (b).

For any $g \in G$,

$$\Lambda(\theta)(gx_i) = \Lambda(\theta)\Big(\sum_{j\in I} r_{ji}(g)x_j\Big)$$

$$= \sum_{j,k\in I} \lambda(\theta_{kj})r_{ji}(g)x_k \quad \text{where } r_{ji} \text{ are defined by}$$

$$gx_i = \sum_{j\in I} r_{ji}(g)x_j \quad .$$

$$g\Delta(\theta)(x_i) = g\sum_{j\in I}\lambda(\theta_{ji})x_j = \sum_{k,j\in I}\lambda(\theta_{ji})r_{kj}(g)x_k.$$

Hence we must prove

(2) $\sum_{j\in I}\lambda(\theta_{kj})r_{ji}(g) = \sum_{j\in I}\lambda(\theta_{ji})r_{kj}(g)$ for all i,k,g.

Now θ is a G map, hence

$$g\theta(v\otimes x_i) = \theta(gv\otimes gx_i).$$

Thus $g\sum_{j\in I}\theta_{ji}(v)\otimes x_j = \sum_{j,k\in I} g\,\theta_{ji}(v)r_{kj}(g)\otimes x_k$

$$= \theta\Big(\sum_{j\in I} r_{ji}(g)\ gv\otimes x_j\Big) = \sum_{j,k\in I} r_{ji}(g)\,\theta_{kj}(gv)\otimes x_k.$$

Hence for all $v\in V$,

$$\sum_{j\in I} g\,\theta_{ji}(v)r_{kj}(g) = \sum_{j\in I} r_{ji}(g)\theta_{kj}(gv),$$

i.e. $$\sum_{j\in I}\theta_{ji}r_{kj}(g) = \sum_{j\in I} r_{ji}(g)\,\theta_{kj}{}^{g}.$$

Applying λ to the above and using (1) we get (2).

Now Δ is a K algebra map and

$$\ker\Delta = \{\theta \mid \theta_{ij}\in J(D) \text{ for all } i,j\in I\}.$$

Clearly $\ker\Delta$ is locally nilpotent (see (B) definition of 1.2 for locally nilpotent) and so since F is local, E is local. Thus E only has the idempotents 0 and 1, i.e. $V\otimes W$ is indecomposable.

Note When G is finite the above theorem says that if H is a normal subgroup of G, V,W finite dimensional indecomposable

G modules such that $V|_H$ is indecomposable and H acts trivially on W then $V \otimes W$ is indecomposable.

It now follows that $St \otimes I(\lambda)^{Fr}$ is indecomposable by taking G to be the simply connected Chevalley group over K, an algebraically closed field of characteristic $p > 0$, $J = \mathfrak{m}^{[p]}$ (so that $R_J = R^{(p)}$), $V = St$ and $W = I(\lambda)^{Fr}$.

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Endomorphism Rings and Induction over a Unipotent Quotient

Introduction

Let K be an algebraically closed field of characteristic $p > 0$, G a finite group and N a normal subgroup such that G/N is a p group. It is a well known theorem of Green [5] that if M is a finite dimensional indecomposable KN module then $\mathrm{Ind}_N^G(M)$ is indecomposable.

The work in this part was motivated by a desire to find a generalisation of this in the context of algebraic groups and their rational representations. The main theorem (3.1) is such a generalisation; it is not, however, as good as we can conjecture.

In order to prove the main theorem we develop, in section 1, results on the endomorphism ring of a comodule of finite width which may be of independent interest.

1 The endomorphism ring of a comodule of finite width

1.1

Let (R,μ,ε) be a coalgebra over K, an algebraically closed field. By a comodule we shall mean a left comodule.

Definition An R comodule V is uniformly bounded if for some $n \in \mathbb{N}$, there is a short exact sequence

$$0 \to V \to \sum_{i=1}^{n} \oplus R \quad \text{of R comodules.}$$

Lemma Let V be an R comodule, $\{S_\gamma\}_{\gamma\in \underline{B}}$ a full set of simple R comodules and

V_γ = the sum of all simple subcomodules of V isomorphic to S_γ, for each $\gamma \in \underline{B}$. The following are equivalent :

(a) V is uniformly bounded ;

(b) $\mathrm{soc}_R(V)$ is uniformly bounded ;

(c) the injective envelope of V is uniformly bounded ;

(d) there exists an $N \in \mathbb{N}$ such that

$$\dim V \ / \ (\dim S_\gamma)^2 \leqslant N \text{ for all } \gamma \in \underline{B} .$$

Proof (a) $\Rightarrow$ (b) clear.

(b) $\Rightarrow$ (a). Using the injectivity of $\sum_{i=1}^{n} \oplus R$ we may complete

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{soc}_R(V) \arrow[r, "\mathrm{inc.}"] \arrow[d, "\alpha"'] & V \arrow[dl, "\hat{\alpha}"] \\ & \sum_{i=1}^{n} \oplus R & \end{tikzcd}$$

to a commutative diagram via an R morphism $\hat{\alpha}$. Here α, n have been chosen so that

$$0 \to soc_R(V) \xrightarrow{\tilde{\alpha}} \sum_{i=1}^{n} \oplus R$$ is a short exact

sequence of R comodules.

Now $\ker \tilde{\alpha} \cap soc_R(V) = \ker \alpha = 0$. Hence $\tilde{\alpha}$ is a monomorphism and V is uniformly bounded.

Let I(V) be an injective envelope of V, then (b)$\Leftrightarrow$(c) is clear since $soc_R(I(V)) \cong soc_R(V)$.

(b)$\Rightarrow$(d)

There exists a short exact sequence

$$0 \to soc_R(V) \to \sum_{i=1}^{n} \oplus R$$ for some n, hence

there is a short exact sequence

$$0 \to \sum_{\gamma \in \underline{B}} \oplus V_\gamma \to \sum_{i=1}^{n} \oplus soc_R(R).$$

Now by the algebraic closure of K and the structure theorem

[6, 1.5g] , $$soc_R(R) \cong \sum_{\gamma \in \underline{B}} \oplus \left(\sum_{j=1,\ldots \dim S_\gamma} \oplus S_\gamma \right).$$

It follows that for each $\gamma \in \underline{B}$ we have the exact sequence

$$0 \to V_\gamma \to \sum_{i=1}^{n} \oplus \left(\sum_{j=1,\ldots \dim S} \oplus S_\gamma \right)$$

and so $\dim V_\gamma \leqslant n (\dim S_\gamma)^2$ for all γ.

(d)$\Rightarrow$(b) Reverse the steps of (b)$\Rightarrow$(d).

<u>Definition</u> (see [6. 2.3]) An R comodule V has finite width if for each simple R comodule S_γ, S_γ occurs only finitely many times in $soc_R(V)$.

Definition If $\{S_\gamma\}_{\in \underline{B}}$ is a full set of simple R comodules, W an R comodule and π a subset of $\underline{B}$ then

$O_\pi(W)$ = the unique maximal R subcomodule of W each of whose composition factors is isomorphic to S_γ for some $\gamma \in \pi$. For a comodule M we put $\pi(M) = \{\gamma \in \underline{B} \mid S_\gamma$ is a composition factor of M$\}$.

Remark It is clear from the definition that an R comodule V has finite width if and only if $\dim \mathrm{Hom}_R(S_\gamma, V) < \infty$ for all $\gamma \in \underline{B}$. We now show that this is equivalent to an apparently stronger condition.

Proposition V, an R comodule, has finite width if and only if for any finite dimensional R comodule M, $\dim \mathrm{Hom}_R(M, V) < \infty$.

Proof ($\Leftarrow$) is evident.

($\Rightarrow$) Let M be a finite dimensional R comodule. Clearly $\mathrm{Hom}_R(M,V)$ may be identified with $\mathrm{Hom}_R(M, O_{\pi(M)}(V))$. $\mathrm{Soc}_R(O_{\pi(M)}(V))$ involves only finitely many simples each with a finite multiplicity. Hence $O_{\pi(M)}(V)$ is uniformly bounded by the above lemma. Thus there exists an exact sequence

$$0 \to O_{\pi(M)}(V) \to \sum_{i=1}^{n} \oplus R \quad \text{for some } n \in \mathbb{N}.$$

Hence $\mathrm{Hom}_R(M,V)$ may be identified with a subspace of

$$\mathrm{Hom}_R(M, \sum_{i=1}^{n} \oplus R) \cong \sum_{i=1}^{n} \oplus \mathrm{Hom}_R(M,R).$$

Thus it is enough to prove that $\mathrm{Hom}_R(M,R)$ is finite dimensional.

Regarding $\mathcal{E}: R \to K$ as a coalgebra map we get $R \cong \mathcal{E}^{o}(K)$ by 2.2e of part A. Hence by 2.2c of part A

$\mathrm{Hom}_R(M,R) \cong \mathrm{Hom}_K(\mathcal{E}_o(M),K)$ as K spaces. The latter is finite dimensional and hence so is the former. This completes the proof of the proposition.

Definition Let V be an R comodule, V_1 an R subcomodule of V. We say V_1 is fully invariant in V if for each $\theta \in \mathrm{End}_R(V)$

$$\theta(V_1) \leq V_1.$$

Definition Let V be an R comodule, say V is locally fully invariant if it is the union of its finite dimensional fully invariant subcomodules.

Corollary 1 If V is an R comodule of finite width then V is locally fully invariant.

Proof Let $M \leq V$, M a finite dimensional subcomodule. Put $E = \mathrm{End}_R(V)$, $H = \mathrm{Hom}_R(M,V)$. H is finite dimensional by the proposition, hence $M_o = \sum_{\alpha \in H} \alpha(M)$ is finite dimensional. $E.M \leq M_o$ hence E.M is a finite dimensional fully invariant subcomodule containing M. It follows that V is locally fully invariant.

Corollary 2 Let $(S,\mu_S,\mathcal{E}_S)$ be a coalgebra over K and let $\varphi: R \to S$ be a morphism of coalgebras. If W is an S comodule of finite width then $\varphi^{o}(W)$ is an R comodule of finite width.

Proof Let X be any finite dimensional R comodule, then $\mathrm{Hom}_R(X, \varphi^0(W)) \cong \mathrm{Hom}_S(\varphi_0(X), W)$ by 2.2c of part A. By the proposition, the latter and hence the former is finite dimensional. Thus $\varphi^0(W)$ has finite width.

Remarks 1 If R is any coalgebra, $V = \sum_{i \in \mathbb{N}} \oplus R$ does not have finite width and is not locally fully invariant.

To see that V is not locally fully invariant observe that $E = \mathrm{End}_R(V)$ contains a copy of the restricted symmetric group on $\mathbb{N}$, $\mathrm{Resymm}(\mathbb{N})$. $\sigma \in \mathrm{Resymm}(\mathbb{N})$ acts on V by

$$\sigma(r_1, r_2, \ldots r_i, \ldots) = (r_{\sigma(1)}, r_{\sigma(2)}, \ldots, r_{\sigma(i)}, \ldots)$$

where each $r_j \in R$ and almost all r_j are zero. It is easy to see that if $0 \neq v \in V$ then the $\mathrm{Resymm}(\mathbb{N})$ orbit of v lies in no finite dimensional subspace of V. Hence V has no nonzero fully invariant, finite dimensional subspaces.

2 The class of R comodules of finite width is closed under taking submodules, but not quotients. Let K be an algebraically closed field of characteristic $p > 0$. Put R = the Hopf algebra $K[X]$, with structure maps μ, ε determined by

$$\mu(X) = 1 \otimes X + X \otimes 1 \quad \text{and} \quad \varepsilon(X) = 0.$$

R viewed as a left R comodule has finite width (in fact $\mathrm{soc}_R(R) = K$), but $\mathrm{soc}_R(R/K) = V/K$ where $V = K + \sum_{i \geq 0} KX^{p^i}$. Every simple R comodule is isomorphic to K hence V/K does not have finite width.

1.2

In 1.2 and 1.3 $(R,\mu,\mathcal{E})$ is a coalgebra over K algebraically closed, V is an R comodule of finite width, and

$\mathcal{A}$ = the set of fully invariant finite dimensional subcomodules of V. For $U \in \mathcal{A}$ we define the map

$\rho_U : \mathrm{End}_R(V) \to \mathrm{End}_R(U)$ to be restriction and put

$$E = \mathrm{End}_R(V),\ E_U = \rho_U(E).$$

Proposition 1 (Local Fitting Lemma) For any $\alpha \in E$,

$$V = \left(\bigcup_{n=1}^{\infty} \ker \alpha^n\right) \oplus \left(\bigcup_{U\in\mathcal{A}} \bigcap_{n=1}^{\infty} \alpha^n(U)\right).$$

Proof Put

$$X = \bigcup_{n=1}^{\infty} \ker\alpha^n,\quad Y = \bigcup_{U\in\mathcal{A}} \bigcap_{n=1}^{\infty} \alpha^n(U).$$

Let $v \in X \cap Y$. Then $v \in \bigcap_{n\geqslant 1} \alpha^n(U)$ for some $U \in \mathcal{A}$.

Now U is finite dimensional so there exists an $N > 0$ such that $\alpha^{N+s}(U) = \alpha^N(U)$ for all $s \geqslant 0$, clearly since v is in X we may pick N so that $v \in \ker\alpha^N$. Thus $v = \alpha^N(u)$ for some $u \in U$ and $\alpha^N(v) = 0$.

But $\beta = \alpha^N \big|_{\alpha^N(U)} : \alpha^N(U) \to \alpha^N(U)$ is an isomorphism since β is an epimorphism and $\alpha^N(U)$ is finite dimensional. Hence $\alpha^N(v) = 0$ implies $v = 0$. Thus $X \cap Y = 0$.

Now if $v \in V$, $v \in U_1$ for some $U_1 \in \mathcal{A}$ since V is locally fully invariant. There exists an $M > 0$ such that $\alpha^{M+s}(U_1)$ $= \alpha^M(U_1)$ for all $s \geqslant 0$.

$$= \alpha^{2M}(v_1) \text{ for some } v_1 \in U_1 \text{ and hence}$$

$v - \alpha^M(v_1) \in \ker\alpha^M \subseteq X$. Also $\alpha^M(v_1) \in \bigcap_{n\geq 1} \alpha^n(U_1) \subseteq Y$.

$v = v - \alpha^M(v_1) + \alpha^M(v_1) \in X \oplus Y$ so $V = X \oplus Y$.

<u>Definition</u> Say an ideal I of E is locally nilpotent if for each $U \in \mathcal{A}$ there is an integer n_U such that $I^{n_U}(U) = 0$.

<u>Lemma</u> If I and J are locally nilpotent ideals of E, then I + J is locally nilpotent.

<u>Proof</u> For $U \in \mathcal{A}$ there exist n_U and m_U such that $I^{n_U}(U) = J^{m_U}(U) = 0$ and we get $I^{n_U + m_U - 1}(U) = 0$.

<u>Definition</u> $N_L(E)$ = the sum of all locally nilpotent ideals of E.

<u>Proposition 2</u> $N_L(E)$ = the Jacobson radical of E and is locally nilpotent.

<u>Proof</u> Note from the Local Fitting Lemma we have $\alpha \in E$ is a unit if and only if $\ker\alpha = 0.$, since if $\ker\alpha = 0$,
$V = \bigcup_{U\in\mathcal{A}} \bigcap_{n\geq 1} \alpha^n(U)$ showing α is onto.

Let $\beta \in N_L(E)$ and $v \in \ker(1+\beta)$, $v \in U$ say for some $U \in \mathcal{A}$. $\beta|_U$ is nilpotent and so $(1+\beta)|_U$ is a unit, hence $v = 0$. Thus $\ker(1+\beta) = 0$, $1+\beta$ is a unit and β is quasiregular. Thus $N_L(E)$ is a quasiregular ideal and so $N_L(E) \subseteq J(E)$, the Jacobson radical.

Let $W \in \mathcal{A}$. W is finite dimensional so $J(E)^m W = J(E)^{m+1} W$ for some $m \in \mathbb{N}$. Hence by Nakayama's lemma $J(E)^m W = 0$ and so

$J(E)$ is locally nilpotent. Hence $N_L(E) = J(E)$ and is locally nilpotent as required.

Corollary $\bigcap_{n=1}^{\infty} J(E)^n = 0$.

Proof Clearly $\rho_U(J(E)) \subseteq J(E_U)$ for any $U \in \mathcal{S}$ and so

$\rho_U(\bigcap_{n=1}^{\infty} J(E)^n) \leqslant \bigcap_{n=1}^{\infty} J(E_U)^n = 0$ since $J(E_U)$ is a nilpotent ideal of the finite dimensional algebra E_U. But $V = \bigcup_{U \in \mathcal{S}} U$

and so $\bigcap_{n=1}^{\infty} J(E)^n$ acts trivially on V, hence $\bigcap_{n=1}^{\infty} J(E)^n = 0$.

Proposition 3 $J(E) = \bigcap_{U\in} \rho_U^{-1}(J(E_U))$.

Proof Similar to the proof of the corollary.

Example $R = K[X]$ the Hopf algebra over K, an algebraically closed field of characteristic zero with structure maps μ, ε determined by $\mu(X) = 1 \otimes X + X \otimes 1$, $\varepsilon(X) = 0$.

Put $V = R$ (considered as a left comodule) and $N = \{\theta \in E \mid \theta(1) = 0\}$.

Since $\mathrm{soc}_R(V) = K$ we get that for any finite dimensional subcomodule W of V and $\theta \in N$, $\dim \theta(W) < \dim W$ (since $K \leqslant \ker\theta \cap W$). It follows that if $\dim W = n$ and $\theta_1, \ldots \theta_n \in N$ then $\theta_1 \theta_2 \ldots \theta_n (W) = 0$ and so N is locally nilpotent. Clearly N has codimension 1 so $N = J(E)$.

Let $D : K[X] \to K[X]$ be the differentiation map. One may easily see that $D \in N$ and $D^n \neq 0$ for all $n \in \mathbb{N}$. Hence $J(E)^n \neq 0$ for all $n \in \mathbb{N}$ in fact $J(E)$ is not nil.

1.3

Lemma 1 If $a + J(E)$ is an idempotent in $E/J(E)$ then there exists an $e = e^2 \in E$ such that $a + J(E) = e + J(E)$.

Proof Put $a_1 = a$, $x_1 = a_1{}^2 - a_1$ and for $n > 1$ define a_n, x_n inductively by $a_n = a_{n-1} + x_{n-1} - 2x_{n-1}a_{n-1}$, $x_n = a_{n-1}{}^2 - a_{n-1}$. It is easily verified that $x_n \in J(E)^{2^n}$ and so if $U \in \mathcal{S}$, $x_m(U) = 0$ for all $m \geqslant N_U$ say. We define $e = a_\infty$ at $U \in \mathcal{S}$ by

$a_\infty(u) = a_{N_U}(u)$ for $u \in U$. Clearly this defines an R endomorphism of V and $a_\infty{}^2 = a_\infty$. It remains to prove that $a_\infty - a \in J(E)$.

For $U \in \mathcal{S}$, $\rho_U(a_\infty - a) = \rho_U(a_{N_U} - a)$. But $a_{N_U} - a \in J(E)$ hence $\rho_U(a_\infty - a) \in J(E_U)$. It follows from proposition 3 of 1.2 that $a_\infty - a \in J(E)$ as required.

Remark The above procedure for producing idempotents is, of course, a limiting version of the well known Brauer idempotent lifting process.

Lemma 2 (cf [4, Theorem 44.3 (2)]) If $1 = f_1 + \ldots\ldots + f_n$ is an orthogonal decomposition of 1 in $E/J(E)$ there is an orthogonal decomposition $1 = e_1 + \ldots\ldots + e_n$ of 1 in E such that $e_i + J(E) = f_i$ for all $1 \leqslant i \leqslant n$.

Proof By induction on n, $n = 2$ being the preceding lemma. Suppose $n > 2$ and the lemma is true for $n - 1$. We may choose a lift $\{e, e_3, \ldots e_n\}$ of $\{f_1 + f_2, f_3, \ldots f_n\}$. Let $b + J(E) = f_1$, put $a = ebe$ then

$$a + J(E) = ebe + J(E) = (f_1 + f_2)f_1(f_1 + f_2) = f_1.$$

It follows that if a_∞ is defined as in the proof of lemma 1 $a_\infty e = ea_\infty = a_\infty$ (since a_∞ is defined at a point by a polynomial in a). Put $e_1 = a_\infty$ then

$$\{e_1, e - e_1, e_3, \ldots e_n\} \text{ is a lift of } \{f_1, \ldots f_n\}.$$

Proposition 1 V is indecomposable if and only if $E/J(E) \cong K$.

Proof ($\Rightarrow$) We first show that E_U is local for all $U \in \mathcal{A}$. Suppose $\alpha \in E$ and $\rho_U(\alpha)$ is an idempotent in E_U. By the Local Fitting Lemma $\bigcup_{n>1} \ker \alpha^n = 0$ or V. Hence α is locally nilpotent or a unit which implies $\rho_U(\alpha)$ is nilpotent or a unit. Hence $\rho_U(\alpha) = 0$ or 1. Hence $E_U/J(E_U) \cong K$ for all $U \in \mathcal{A}$. Now if $\theta \in E$, $U \in \mathcal{A}$ there exists a unique $\lambda_U \in K$ such that $\theta - \lambda_U$ is nilpotent on U. If W is also an element of $\mathcal{A}$ there exists a unique $\lambda_W \in K$ such that $\theta - \lambda_W$ is nilpotent on W and a unique $\lambda_{U+W} \in K$ such that $\theta - \lambda_{U+W}$ is nilpotent on $U + W$. Now since $\theta - \lambda_{U+W}$ is nilpotent on U, $\lambda_{U+W} = \lambda_U$, similarly $\lambda_{U+W} = \lambda_W$ hence $\lambda_U = \lambda_W$. Thus there is a unique $\lambda \in K$ such that $\rho_W(\theta - \lambda) \in J(E_W)$ for all $W \in \mathcal{A}$. Hence by proposition 3 of 1.2, $\rho - \lambda \in J(E)$, showing that $J(E)$ has codimension 1, i.e. $E/J(E) \cong K$.

($\Leftarrow$) Let $\pi : E \to E/J(E)$ denote the natural map. If $e = e^2 \in E$, then $E/J(E) \cong K$ implies $\pi(e) = J(E)$ or $1 + J(E)$. Hence e or $1 - e \in J(E)$, but e and $1 - e$ are idempotents so e or $1 - e \in \bigcap_{n \geq 1} J(E)^n = 0$ (corollary to proposition 2 of 1.2). Thus e or $1 - e = 0$, E has only one nonzero idempotent and V is indecomposable.

<u>Proposition 2</u> $E/J(E)$ is finite dimensional if and only if there exists an R comodule decomposition

$V = V_1 \oplus \ldots\ldots \oplus V_n$ for some $n \in \mathbb{N}$, such that each V_i is indecomposable.

<u>Proof</u> ($\Rightarrow$) Let $1 = f_1 + \ldots + f_n$ be an orthogonal decomposition of 1 in $E/J(E)$ such that each f_i is primitive. Lift to an orthogonal decomposition $1 = e_1 + \ldots.+e_n$ of 1 in E by lemma 2 of 1.3.

$V = e_1V \oplus \ldots\ldots \oplus e_nV$ and each e_iV is indecomposable.

($\Leftarrow$) Let $V = V_1 \oplus \ldots\ldots \oplus V_n$ be an R comodule decomposition of V such that each V_i is indecomposable.

Put $I = \{1, \ldots n\}$. We identify $\alpha \in \mathrm{End}_R(V) = E$ with a matrix $(\alpha_{ij})_{i,j \in I}$ where $\alpha_{ij} \in \mathrm{Hom}_R(V_j, V_i)$. Note that each $\mathrm{End}_R(V_i)$ is local.. Now by Fitting's theorem

$J(\mathrm{End}_R(V)) = \{ \alpha \mid$ each α_{ij} is not an isomorphism$\}$.

Choose a decomposition $I = \dot{\bigcup}_{t=1,\ldots r} I_t$ of I such that i and j are in the same I_t if and only if $V_i \cong V_j$. Renumber the V_i's, if necessary, so that $1,2,\ldots n_1 \in I_1$, $n_1+1,\ldots n_2 \in I_2$ etcetera. Put $X_i = V_{n_i}$ for $i = 1,2,\ldots r$ and put

$D_i = \mathrm{End}\, X_i \;/\; J(\mathrm{End}\, X_i) \cong K$ by the above proposition.

Say an element $\beta \in \mathrm{Hom}_R(V_j, V_i)$ is singular if β is not an isomorphism. We define a map $\overline{\psi} : \mathrm{End}_R(V) \to W$ by $\overline{\psi}((\alpha_{ij})) = (\alpha_{ij} +$ the set of singular elements of $\mathrm{Hom}_R(V_j, V_i))$ where (α_{ij}) is the matrix representing an R map α and W is a matrix ring identifiable with $\prod_{t=1}^{r} (D_t)_{|I_t|}$. The kernel of $\overline{\psi}$ is $J(E)$ hence $E/J(E)$ is finite dimensional.

<u>Note</u> This proposition together with lemma 2 of 1.3 shows that if V has a decomposition $V = V_1 \oplus \ldots \oplus V_n$ such that each V_i is indecomposable then E is semiperfect in the sense of [1].

2 Algebra automorphisms of a finite dimensional simple algebra

Let E be a finite dimensional simple algebra over K, an algebraically closed field. Let E_0 denote the set of units in E. E_0 is an algebraic subgroup of $GL_K(E)$. E_0 is isomorphic to $GL_n(K)$ where $E \cong M_n(K)$.

We put E_1 = the elements of E_0 having determinant 1. $E_1 \cong SL_n(K)$.

Now there is a map $\theta : E_1 \to GL(E)$ given by $\theta(x)a = xax^{-1}$ for $x \in E_1$, $a \in E$. It is easy to check that is a morphism of algebraic groups. This gives rise naturally to a map $d\theta : \mathcal{L}(E_1) \to \mathcal{L}(GL(E))$ where $\mathcal{L}(E_1)$ and $\mathcal{L}(GL(E))$ denote the Lie algebras of E_1 and $GL(E)$ respectively. It is our first aim to find $\ker d\theta$.

We identify E with $\mathrm{End}_K(V)$ for some K space V. Let $\{v_1, \ldots, v_n\}$ be a basis of V and for $1 \leq i,j \leq n$ let e_{ij} be the element of E defined by $e_{ij}(v_l) = \delta_{lj} v_i$ for $1 \leq l \leq n$. We define functions $Z_{\alpha\beta,ij} : GL_K(E) \to K$ for $1 \leq \alpha,\beta,i,j \leq n$ by
$$ge_{ij} = \sum_{1 \leq \alpha,\beta \leq n} Z_{\alpha\beta,ij}(g)e_{\alpha\beta} \quad \text{for any } g \in GL(E).$$

Define functions $X_{ij} : E_1 \to K$ by
$$x = \sum_{1 \leq i,j \leq n} X_{ij}(x)e_{ij} \quad \text{for } x \in E_1.$$

Let $\Delta : GL(E) \to K$ be the inverse determinant function, then

$K[GL_K(E)] = K[Z_{\alpha\beta,ij}, \Delta \mid 1 \leq \alpha,\beta,i,j \leq n]$ and

$K[E_1] = K[X_{ij} \mid 1 \leq i,j \leq n]$.

Θ gives rise to a comorphism $\Theta^* : K[GL(E)] \to K[E_1]$;

$\Theta^*(Z_{\alpha\beta,ij})(x) = Z_{\alpha\beta,ij}(\Theta(x))$, for any $g \in E_1$.

Now $\Theta(x) \circ e_{ij} = xe_{ij}x^{-1} = \sum_{\alpha,\beta} X_{\alpha\beta}(x)e_{\alpha\beta}e_{ij}\, x^{-1}$

$= \sum_{\alpha} X_{\alpha i}(x)e_{\alpha j}x^{-1} = \sum_{\alpha,r,s} X_{\alpha i}(x)e_{\alpha j}Y_{rs}(x)e_{rs}$

where $Y_{rs} : E_1 \to K$ is defined by $x^{-1} = \sum_{1 \leq r,s \leq n} Y_{rs}(x)e_{rs}$.

Hence $\Theta(x) \circ e_{ij} = \sum_{\alpha,\beta} X_{\alpha i}(x)e_{\alpha\beta}\, Y_{j\beta}(x)$ for any $x \in E_1$

and so

$$\Theta^*(Z_{\alpha\beta,ij}) = X_{\alpha i}Y_{j\beta} \quad .$$

Suppose $\gamma \in \ker d\Theta$, then

$$\gamma\,(X_{\alpha i}Y_{j\beta}) = 0 \text{ for all } 1 \leq i,j,\alpha,\beta \leq n.$$

γ is a derivation so $0 = \gamma\,(X_{\alpha i}Y_{j\beta})$

$= \varepsilon(X_{\alpha i})\gamma\,(Y_{j\beta}) + \gamma(X_{\alpha i})\varepsilon(Y_{j\beta}) = \delta_{\alpha i}\,\gamma(Y_{j\beta}) + \delta_{j\beta}\gamma\,(X_{\alpha i})$.

Thus for $\alpha \neq i$, $j = \beta$ we get $\gamma(X_{\alpha i}) = 0$.

For $\alpha = i$, $j = \beta$ we get $\gamma\,(Y_{\beta\beta}) + \gamma\,(X_{\alpha\alpha}) = 0$.

Hence $\gamma\,(X_{11}) = \ldots\ldots = \gamma\,(X_{nn})$. Since γ is a derivation and $\det(\,(X_{ij})) = 1$ we get $\gamma\,(\det(\,(X_{ij})) = 0$.

Expanding this relation we obtain

$$\gamma\,(X_{11}) + \ldots\ldots + \gamma\,(X_{nn}) = 0.$$

We can now describe the kernel of $d\Theta$.

(a) If char $K = p > 0$ and $p \nmid n$ or char $K = 0$ then $\ker d\theta = 0$.

(b) If char $K = p > 0$ and $p \mid n$ then $\ker d\theta$ is the K span of the derivation D defined on the generators by

$$D(X_{ij}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Since D acts as ε on the generators X_{ij}, we have $D^g = \varepsilon_g D \varepsilon_{g^{-1}} = D$ for all $g \in E_1$.

Let $U_1 = \{ x \in E_1 \mid X_{ij}(x) = \begin{cases} 1 & i = j \\ 0 & i > j \end{cases} \}$

$\mathcal{L}(U_1)$ may be identified with

$$\{ \gamma \in \mathcal{L}(E_1) \mid \gamma(X_{ij}) = 0 \quad \text{for } i \geqslant j \}.$$

Now if U is any unipotent subgroup of E_1, U may be put into upper triangular form, i.e. there is a $g \in E_1$ such that $U^g \subseteq U_1$. Now $(KD \cap \mathcal{L}(U))^g = KD^g \cap \mathcal{L}(U^g)$ $\subseteq KD \cap \mathcal{L}(U_1) = 0$. Hence for any unipotent subgroup U of E_1 $d\theta|_{\mathcal{L}(U)} : \mathcal{L}(U) \to \mathcal{L}(GL_K(E))$ is injective.

It can be checked that the set of K algebra automorphisms of E form a closed subgroup of $GL(E)$. We put $G = \mathrm{Aut}_{K\ alg}(E)$. Clearly $\mathrm{Im}\,\theta \subseteq G$ and by Theorem 2.35 of [7] it is onto. Let H be a closed connected unipotent subgroup of G, put $L = \theta^{-1}(H)$. $\theta(L^o)$ has finite index in H hence $\theta(L^o) = H$.

Now as abstract groups $L/\ker\theta \cong \operatorname{Im}\theta = H$.

$$\operatorname{Ker}\theta = \left\{ \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} \;\middle|\; \lambda^n = 1 \right\} = \text{centre of } E_1 ,$$

hence L is soluble.

Thus L^o is a soluble connected algebraic group hence $L^o = T.L_u$, the semi direct product of a maximal torus T and the unipotent radical of L, L_u. Now $\theta(T)$ is a torus in H which is unipotent, hence $\theta(T) = 1$. But $\ker\theta$ is finite and T is connected so $T = 1$. Thus L^o is unipotent.

Clearly $\theta : L^o \to H$ is an isomorphism of abstract groups, a morphism of varieties and separable since

$$d\theta\,|_{\mathcal{L}(L^o)} : \mathcal{L}(L^o) \to \mathcal{L}(H) \text{ is injective and } \dim L^o = \dim H.$$

Thus $\theta\,|_{L^o} : L^o \to H$ is an isomorphism of algebraic groups.

Let $\psi : H \to L^o$ be the inverse. The discussion so far shows the following.

<u>Proposition</u> Let E be a finite dimensional simple algebra over K, an algebraically closed field. Let E_o be the group of units in E considered as an algebraic group. Let H be a unipotent algebraic group over K acting rationally as a group of K algebra automorphisms of E giving rise to the map $\xi : H \to \operatorname{Aut}_{K\ alg}(E)$. There exists a morphism of algebraic groups $\psi : H \to E_1$, the elements of E of determinant 1, such that $\psi(h)a\,\psi(h)^{-1} = \xi(h)a$ for all $h \in H, a \in E$.

<u>Remark</u> Clearly ψ is uniquely determined.

<u>A Digression</u> Let G be a simply connected algebraic Chevalley group over K, algebraically closed of characteristic $p > 0$, and let $\underset{\sim}{u}_1$ be the restricted enveloping algebra of the Lie algebra of G, realised inside $K[G]^*$. If V is a simple $\underset{\sim}{u}_1$ module we may use the above procedure to give V a rational G module structure.

Put I = the annihilator of V in $\underset{\sim}{u}_1$, an ideal. $J(\underset{\sim}{u}_1)$ = the Jacobson radical of $\underset{\sim}{u}_1$ and $B = \underset{\sim}{u}_1 / J(\underset{\sim}{u}_1)$. Let

$1 = e_1 + \ldots\ldots + e_n$ be an orthogonal decomposition of 1 in B, into centrally primitive idempotents. G acts rationally on $\underset{\sim}{u}_1$ by conjugation, i.e. $a^g = \varepsilon_g a\, \varepsilon_{g^{-1}}$ and hence on B. G permutes the finite set of idempotents $\{e_1, \ldots\ldots e_n\}$ and since G is connected it must fix each e_i.

Now $A/I \cong e_i B$ for some i. Put $e = e_i$. $E = eB$ is a simple algebra isomorphic to $\mathrm{End}_K(V)$ on which G acts as a group of K algebra automorphisms.

Let U and U' be the standard maximal unipotent subgroups of G, arising from the Chevalley construction. Let $\pi : eB \to \mathrm{End}_K(V)$ be the natural map.

Now U acts on eB by K algebra automorphisms so by the proposition there is a map $\sigma : U \to eB$ such that $(eb)^x = \sigma(x)\pi(eb)\sigma(x^{-1})$ for all $x \in U$, $b \in B$.

Similarly we obtain a map $\sigma' : U' \to eB$ and σ and σ' are uniquely determined. We also know that V has the structure of a simple rational G module ([3, 6.6]) via ρ say, in such a way that ρ on restriction gives rise to th natural map $\underset{\sim}{u}_1 \to \mathrm{End}_K(V)$. Now for $x \in U$ we have $\rho(x)\,\pi(eb)\,\rho(x^{-1}) = (eb)^x$ and so $\rho|_U = \sigma$, similarly $\rho|_{U'} = \sigma'$. Since G is generated by U and U' ρ is completely determined on G by σ and σ'.

3 Induction over a unipotent quotient

3.1

We make the following conjecture.

__Conjecture__ Let G be an algebraic group over K, an algebraically closed field. Let H be a closed normal subgroup and suppose G/H is unipotent. Let $\varphi : K[G] \to K[H]$ be the restriction map. If V is an indecomposable rational KH module of finite width then $\varphi^{0}(V)$ is an indecomposable KG module of finite width.

__Remark__ By corollary 2 of the proposition of 1.1 $\varphi^{0}(V)$ has finite width. We cannot prove the conjecture but we prove a restricted version in this section.

__Definition__ Let G be an algebraic group over K and let H be a closed subgroup. We say a rational KH module V is a finite G component if there is a rational G module W such that

$\varphi_0(W) \cong Y_1 \oplus \ldots\ldots \oplus Y_n$ as KH modules, for some $n \in \mathbb{N}$,

where each Y_i is indecomposable of finite width $Y_1 \cong V$ and $\varphi : K[G] \to K[H]$ is the restriction map.

__Theorem__ Let G be an algebraic group over K, algebraically closed, H a closed normal subgroup such that G/H is unipotent. Let $\varphi : K[G] \to K[H]$ be the restriction map. If V is an indecomposable rational KH module and a finite G component then $\varphi^{0}(V)$ is a rational indecomposable G module of finite width.

Proof Claim it is enough to prove

(*) G/H connected implies the theorem is true.

Put $N = HG^o$, $\psi_1 : K[G] \to K[N]$, $\psi_2 : K[N] \to K[H]$, restriction maps. $\varphi = \psi_2\psi_1$ so $\varphi^o(V) \cong \psi_1{}^o(\psi_2{}^o(V))$.

Now HG^o/H is connected so (*) implies $\psi_2{}^o(V)$ is indecomposable of finite width. $|G:HG^o| < \infty$ and G/HG^o is unipotent. Thus if char $K = 0$ then $G = HG^o$ and $\varphi^o(V) = \psi_2{}^o(V)$ is indecomposable of finite width. If char $K = p > 0$ then G/HG^o is a finite p group. It follows from the proof of [5, theorem 8] that $\varphi^o(V) \cong \psi_1{}^o(\psi_2{}^o(V))$ is indecomposable.

We now assume that G/H is connected.

Let W be a rational G module such that $\varphi_o(W) = Y_1 \oplus \dots \oplus Y_n$ where each Y_i is indecomposable of finite width (as an H module) and $Y_1 \cong V$.

Let $D = \mathrm{End}_{KH}(W)$ and $E = \mathrm{End}_{KG}(W \otimes K[G]_{(H)})$ where $K[G]_{(H)} = \{f \in K[G] \mid f\circ h = f \text{ for all } h \in H\}$.

We will show that the length of any orthogonal decomposition of 1 in E is at most n.

Let $\{x_i\}_{i \in I}$ be a basis for $K[G]_{(H)}$. If $\theta \in E$ we may define $\theta_{ij} : W \to W$ by $\theta(w \otimes x_i) = \sum_{j \in I} \theta_{ji}(w) \otimes x_j$.

One can easily check that the θ_{ij} so defined are elements of D.

We wish to show that for a fixed i

(1) $\theta_{ji} \in J(D)$ for all but a finite number of j s.

Here J(D) denotes the Jacobson radical of D.

By proposition 2 of 1.3 $D/J(D)$ is finite dimensional. Let C be a K space complement to $J(D)$ in D. C is finite dimensional so there is a finite dimensional fully invariant H submodule U of W such that the restriction map

$$\rho : D \to \mathrm{End}_{KH}(U) \text{ is injective on } C.$$

It follows that if $D_U = \rho(D)$, ρ induces an isomorphism

$$\bar{\rho} : D/J(D) \to D_U/J(D_U). \qquad (2).$$

Now since $W \otimes K[G]_{(H)} \cong \varphi^0(W)$ has finite width it is locally fully invariant. Hence for fixed i $\theta(U \otimes x_i) \leqslant \bar{U}$, a finite dimensional K space, for all $\theta \in E$.

Thus $\theta(U \otimes x_i) \leqslant \sum_{j \in I_i} U \otimes x_j$ for some finite subset I_i of I.

Hence $\theta_{ji}(U) = 0$ for $j \notin I_i$ and so $\theta_{ji} \in J(D)$ for all $j \notin I_i$ by (2), proving (1).

G acts on D as a group of K algebra automorphisms α^g being defined for $\alpha \in D$, $g \in G$ by

$$\alpha^g(z) = g^{-1}\alpha(gz) \quad \text{for each } z \in W.$$

This gives rise to an action of G on $D/J(D)$. We shall show that this action is rational.

Put $U_1 = KGU$, for $\alpha \in D$, $g \in G$ $\alpha(gU) = g.\alpha^g(U) \leqslant gU$ since U is fully invariant under D.

Since $U \leqslant U_1$ it follows that if $\rho_1 : D \to \mathrm{End}_{KH}(U_1)$ is the restriction map ρ_1 gives rise to an isomorphism

$\tilde{\rho}_1 : D/J(D) \to D_{U_1}/J(D_{U_1})$ where $D_{U_1} = \rho_1(D)$.

Now G acts naturally on $\mathrm{End}_K(U_1)$ by conjugation and with this action $\mathrm{End}_K(U_1) \cong U_1 \otimes U_1^*$ hence $\mathrm{End}_K(U_1)$ is a rational G module. D_{U_1} is a sub G module of $\mathrm{End}_K(U_1)$ hence rational, $D_{U_1}/J(D_{U_1})$ is a quotient of a rational module hence rational finally $\tilde{\rho}_1$ is an isomorphism of G modules hence $D/J(D)$ is a rational G module.

Now by the Wedderburn theorem

$D/J(D) \cong R_1 \oplus \ldots \oplus R_l$ say, a direct sum of rings, where $R_i = \mathrm{End}_K(V_i)$ for finite dimensional K spaces V_i.

H acts trivially by conjugation on D, hence on $D/J(D)$. Thus $D/J(D)$ is a rational G/H module. Clearly G/H permutes the central idempotents of $D/J(D)$, since they form a finite set and G/H is connected G fixes each central idempotent.

Thus G/H acts rationally on each R_i as a group of K algebra automorphisms. G/H is unipotent so we may give each V_i the structure of a rational G/H module by the proposition of 2.

Note that the length of an orthogonal decomposition of 1 in $D/J(D)$ into a sum of primitive idempotents is $\sum_{i=1}^{l} \dim V_i$.

Let $\lambda_s : D \to R_s$ be the natural map for $1 \leq s \leq n$, $F_s = \mathrm{End}_{KG}(V_s \otimes K[G]_{(H)})$ and $F = F_1 \oplus \ldots \oplus F_l$, the direct sum of rings. Let V_s afford the representation of G/H σ_s.

For $1 \leqslant s \leqslant l$ define $\Lambda_s : E \to F_s$ by

$$\Lambda_s(\theta)(v \otimes x_i) = \sum_{j\in I} \lambda_s(\theta_{ji})(v) \otimes x_j \qquad \text{for}$$

$v \in V_s$, $i \in I$, $\theta \in E$. By (1) this is a well defined K space map, we must check that $\Lambda_s(\theta) \in F_s$ for any $\theta \in E$.

$$\Lambda_s(\theta)(g(v \otimes x_i)) = \Lambda_s(\theta)(\sigma_s(g)v \otimes gx_i)$$

$$= \Lambda_s(\theta)\Big(\sum_{j\in I} r_{ji}(g)\,\sigma_s(g)v \otimes x_j\Big)$$ where the r_{ji} are defined by $gx_i = \sum_{j\in I} r_{ji}(g)x_j$.

So $\Lambda_s(\theta)(g(v \otimes x_i)) = \sum_{j,k\in I} r_{ji}(g)\,\lambda_s(\theta_{kj})(\sigma_s(g)v) \otimes x_k$.

$$g\Lambda_s(\theta)(v \otimes x_i) = g\Big(\sum_{j\in I} \lambda_s(\theta_{ji})(v) \otimes x_j\Big)$$

$$= \sum_{j,k\in I} \sigma_s(g)\,\lambda_s(\theta_{ji})(v) \otimes gx_j$$

$$= \sum_{j,k\in I} r_{kj}(g)\,\sigma_s(g)\,\lambda_s(\theta_{ji})(v) \otimes x_k.$$

So we must show that

$$\sum_{j\in I} r_{ji}(g)\,\lambda_s(\theta_{kj})\,\sigma_s(g) = \sum_{j\in I} r_{kj}(g)\,\sigma_s(g)\,\lambda_s(\theta_{ji}),$$

i.e. using the definition of the G action on V_s, that

$$(3) \quad \sum_{j\in I} r_{ji}(g)\,\lambda_s(\theta_{kj}{}^{g}) = \sum_{j\in I} r_{kj}(g)\,\lambda_s(\theta_{ji}).$$

Now θ is a G map so $\theta(g(w \otimes x_i)) = g\,\theta(w \otimes x_i)$ for all $w \in W$, $i \in I$.

$$\theta(g(w \otimes x_i)) = \Big(\sum_{j\in I} r_{ji}(g)gw \otimes x_j\Big)$$

$$= \sum_{j,k\in I} r_{ji}(g)\,\theta_{kj}(gw) \otimes x_k.$$

$$g\,\theta(w \otimes x_i) = g \sum_{j\in I} \theta_{ji}(w) \otimes x_j = \sum_{j\in I} g\,\theta_{ji}(w) \otimes gx_j$$

$$= \sum_{j,k\in I} r_{kj}(g)g\,\theta_{ji}(w) \otimes x_k.$$

Hence $\sum_{j\in I} r_{ji}(g)\,\theta_{kj}{}^{g} = \sum_{j\in I} r_{kj}(g)\,\theta_{ji}$; applying λ_s to this equation we get (3).

One may easily check that Λ_s is an algebra homomorphism. Define $\Lambda : E \to F$ to be $\Lambda_1 \oplus \ldots\ldots \oplus \Lambda_l$.

Now if $\theta \in \ker \Lambda$ then for all i, j $\theta_{ij} \in \ker \lambda_s$ for all s, i.e. $\theta_{ij} \in J(D)$. It is easily seen that

$\{\theta \in E \mid \theta_{ij} \in J(D)$ for all $i, j\}$ is a locally nilpotent ideal of E. Thus $\ker \Lambda$ is locally nilpotent. Now

$F_s = \mathrm{End}_{KG}(V_s \otimes K[G]_{(H)}) \cong \mathrm{End}_{KG}(\phi^{o}(V_s)) \cong \mathrm{End}_{KG}((V_s) \otimes K[G]_{(H)})$

$\cong M_{n_s}(\mathrm{End}_{KG}(K[G]_{(H)}))$ where $n_s = \dim V_s$.

Now $\mathrm{End}_{KG}(K[G]_{(H)})$ is a local ring since G/H is unipotent and so

length of an orthogonal decomposition of 1 in F_s into a sum of primitive idempotents $= \dim V_s$.

Since Λ has a locally nilpotent kernel we have

number of idempotents in an orthogonal decomposition of 1 in E into a sum of primitive idempotents $\leqslant \sum_s \dim V_s =$ number of idempotents in an orthogonal decomposition of 1 in D into a sum of primitive idempotents = n.

Now $\varphi^{o}(W) \cong \sum_{i=1}^{n} \oplus \varphi^{o}(Y_i)$ and H normal in G, $Y_i \neq 0$ implies $\varphi^{o}(Y_i) \neq 0$ (4).

So if $e_i : \varphi^{o}(W) \to \varphi^{o}(W)$ are projection maps such that e_i acts as the identity on $\varphi^{o}(Y_i)$ and zero on $\varphi^{o}(Y_j)$ for $j \neq i$, then $1 = e_1 + \ldots + e_n$ is an orthogonal decomposition of 1 of length n. Thus e_1 must be primitive and so $\varphi^{o}(Y_1) \cong \varphi^{o}(V)$ is indecomposable.

Proof of (4) : Let $0 \neq X$ be a rational module for H, suppose $\varphi^{o}(X) = 0$. Without loss of generality X is simple since φ^{o} is left exact. By [2, theorem 4] X embeds in a rational G module Z which we may also take to be simple. Since G/H is connected and G/H permutes the finite set of H homogeneous components of Z,

$$\varphi_o(Z) \cong \sum{}^{\oplus} X \quad \text{(as H modules)}.$$

Thus $0 \neq \varphi^{o}(Z) \cong Z \otimes K[G]_{(H)} = \sum{}^{\oplus} \varphi^{o}(X)$ and so $\varphi^{o}(X) \neq 0$.

This completes the proof of the theorem.

<u>Example</u> We give an example of a finite dimensional KH module V which is not a finite G component but $\varphi^{o}(V)$ is indecomposable. In order to smooth our way we give a definition and a general lemma.

<u>Definition</u> Let G be an algebraic group over K and H a closed subgroup. Let $\varphi : K[G] \to K[H]$ be the restriction map. We say a rational H module V is a G component if

$$V \mid \varphi_0(W) \quad \text{for some rational } G \text{ module } W.$$

<u>Remark</u> If G is finite then any H module V is a G component since $V \mid \varphi_0(\varphi^0(V))$. In general we have the following.

<u>Lemma</u> Let G be an algebraic group over K, H a closed subgroup and $\varphi : K[G] \to K[H]$ the restriction map. A rational H module H is a G component if and only if the sequence

$$0 \to \ker e \to \varphi^0(V) \xrightarrow{e} V \to 0$$ of H modules, where e is the natural map, is exact and split.

<u>Proof</u> ($\Leftarrow$) Obvious.

($\Rightarrow$) Let $V \mid \varphi_0(W)$, W a rational G module. Let $\alpha : V \to \varphi_0(W)$ and $\beta : \varphi_0(W) \to V$ be H maps such that $\beta\alpha = 1_V$. By the universal mapping property there is a unique G map $\hat{\beta} : W \to \varphi^0(V)$ such that $\beta = e\hat{\beta}$. Now $e(\hat{\beta}\alpha) = e\hat{\beta}\alpha = \beta\alpha = 1_V$. Which implies that e is onto and

$$0 \to \ker e \to \varphi_0(\varphi^0(V)) \xrightarrow{e} V \to 0 \text{ is split.}$$

<u>Remark</u> If V is a finite dimensional rational KH module then V is observable (see [2]) if and only if $e : \varphi^0(V) \to V$ is surjective.

Now let $G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x,y,z \in K \right\}$,

$H = \left\{ \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; y,z \in K \right\}$ K algebraically closed.

Let $\varphi : K[G] \to K[H]$ be the restriction map.

$K[G] = K[X,Y,Z]$ where X, Y, Z are as in example 1.3 of part A.

Put $V = K + K\varphi(Y)$ a rational left KH module.

$\varphi^0(V) = \{1 \otimes a + \varphi(Y) \otimes b \mid 1 \otimes a + h_{y,z}\varphi(Y) \otimes b$

$= 1 \otimes ah_{y,z} + \varphi(Y) \otimes bh_{y,z}$ for all $y,z \in K\}$

$\leqslant (V) \otimes K[G]$. Here $h_{y,z} = \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$.

Now $h_{y,z}\varphi(Y) = y + \varphi(Y)$ so we require

$1 \otimes (a + yb) + \varphi(Y) \otimes b = 1 \otimes ah_{y,z} + \varphi(Y) \otimes bh_{y,z}$.

So $b \in K[G]_{(H)}$ and $ah_{y,z} = a + yb$ for all $y,z \in K$.

$K[G]_{(H)} = K[X]$ so if $a = a(X,Y,Z)$, $a(X,Y,Z)h_{y,z} = a(X,Y,Z) + yb(X)$

for all $y,z \in K$. Using the equations

$Xh_{y,z} = X$, $Yh_{y,z} = y + Y$, and $Zh_{y,z} = z + Z$ we eventually

find $a = a_0(X) + Ya_1(X)$ and $b(X) = a_1(X)$ for some polynomials

a_0 and a_1. Thus we may identify $\varphi^0(V)$ with

$$K[X] + YK[X] \leqslant K[G].$$

G is unipotent hence $\mathrm{soc}_G(K[G]) = K$ and so $\varphi^0(V)$ is

indecomposable as a G module.

We show further that the KH module $\varphi_0(\varphi^0(V))$ is indecomposable. Let θ be a KH endomorphism of $\varphi^0(V)$, θ determines K space endomorphisms of K[X], θ_{ij} for $1 \leqslant i,j \leqslant 2$ defined by

$$\theta(a(X)) = \theta_{11}(a(X)) + Y\,\theta_{21}(a(X))$$

$\theta(Yb(X)) = \theta_{12}(b(X)) + Y\,\theta_{22}(b(X))$ for $a(X)$ and $b(X)$ in $K[X]$.

Using the fact that θ is a KH map we get

$\theta_{21} = 0$, and $\theta_{11} = \theta_{22} = \alpha$ say where $\alpha(Xb(X)) = X\alpha(b(X))$ for all $b(X) \in K[X]$. Hence α is multiplication by $f(X)$ for some $f(X) \in K[X]$. We may represent θ as

$$\begin{pmatrix} \alpha_{f(X)} & \beta \\ 0 & \alpha_{f(X)} \end{pmatrix}$$

where $\alpha_{f(X)}$ and β are K endomorphisms of $K[X]$, $\alpha_{f(X)}$ being multiplication by $f(X)$.

It follows that if θ is an idempotent then $\theta = 0$ or 1.

Hence the KH module $\varphi_0(\varphi^0(V))$ is indecomposable.

By the above lemma V cannot be a G component.

3.2

Let K be a field which is the algebraic closure of its prime subfield. We now show that if G is an algebraic group over K, H a normal closed subgroup such that G/H is unipotent and dim G = 1 then every KH module of finite width is a finite G component.

<u>Lemma</u> If $G = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in K \right\}$, K algebraically closed of characteristic $p > 0$ and

$$H = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in K,\ x^q = x \right\}$$ where q is some power of p, then every KH module extends to a rational KG module.

<u>Proof</u> Let $\varphi : K[G] \to K[H]$ be the restriction map.
$K[G] = K[X]$ where $g = \begin{pmatrix} 1 & X(g) \\ 0 & 1 \end{pmatrix}$ for any $g \in G$.

Put $Y = \varphi(X)$. $K[H] = K\ \mathrm{span}\{1, Y, \ldots Y^{q-1}\}$.
Define $\psi : K[H] \to K[G]$ to be the K map given by

$\psi(Y^i) = X^i$ for $1 \leqslant i \leqslant q-1$. Note ψ is not a Hopf algebra map since $\psi(Y^q) = \psi(Y) = X \neq X^q = \psi(Y)^q$, it is however a coalgebra map.

Thus if V is a KH comodule $\psi_0(V)$ is a $K[G]$ comodule and $\varphi_0(\psi_0(V)) \cong (\varphi\psi)_0(V) \cong V$.

<u>Proposition</u> Let K be a field which is the algebraic closure of its prime subfield and let G be an algebraic group over K. Suppose $G^0 \cong K^+$, the one dimensional unipotent group. If H is a finite subgroup of G then any finite dimensional KH module is a finite G component.

<u>Proof</u> Let V be a finite dimensional KH module.

<u>Step 1</u> Without loss of generality (WLOG) $G^oH = G$.

<u>Proof</u> Put $L = G^oH$. Let $\phi : K[G] \to K[H]$, $\psi : K[G] \to K[L]$ and $\psi' : K[L] \to K[H]$ be restriction maps. Suppose $V \mid \psi'_o(W)$ for some finite dimensional rational KL module W. Now $|G:L| < \infty$ implies $W \mid \psi_o(\psi^o(W))$ and so $V \mid \psi'_o\psi_o(\psi^o(W)) = \phi_o(\psi^o(W))$, $\psi^o(W)$ is finite dimensional and so V is a finite G component.

<u>Step 2</u> WLOG char $K = p > 0$.

<u>Proof</u> Suppose char $K = 0$, $H \cap G^o$ is a finite torsion free group so $H \cap G^o = 1$. $K[H]$ is semisimple so if $\varepsilon' : K[H] \to K$ is evaluation at 1

$$V \mid \varepsilon'^o(\varepsilon'_o(V)) \cong (V) \otimes K[H] \cong (V) \otimes \phi_o(K[G]_{(G^o)}) ,$$

the last isomorphism holds since multiplication $H \times G^o \to G$ is an isomorphism of varieties. Thus V is a finite G component.

We now assume char $K = p > 0$.

<u>Step 3</u> WLOG H is a p group.

<u>Proof</u> If P is a Sylow p subgroup of H, $V \mid \theta^o(\theta_o(V))$ where $\theta : K[H] \to K[P]$ is restriction; this follows from D.G.Higman's theorem (see lemma 51.2 of [4]). Suppose $\theta_o(V) \mid \xi_o(W)$ where $\xi : K[G] \to K[P]$ is the restriction map and W is a finite dimensional rational KG module.

Then

$$(1) \quad V \mid \theta^{o}(\theta_{o}(V)) \mid \theta^{o}(\xi_{o}(W)) \cong \varphi_{o}(\eta^{o}(\eta_{o}(W))$$

where $\eta : K[G] \to K[G^{o}P]$ is the restriction map. The above isomorphism follows from the fact that any crossection for P in H is also a crossection for PG^{o} in G. This in turn is true since

$$|G:PG^{o}| = |HG^{o}:PG^{o}| = \frac{|HG^{o}/G^{o}|}{|PG^{o}:G^{o}|} = \frac{|H/H\cap G^{o}|}{|P/P\cap G^{o}|} = |H:P|$$

because $P\cap G^{o} = H\cap G^{o}$ as $H\cap G^{o}$ is a normal p subgroup of H.

Thus by (1), V is a finite G component.

<u>Step 4</u> $G^{o} \leqslant Z(G)$, the centre of G.

<u>Proof</u> H is a p group hence $G = G^{o}H$ is unipotent hence nilpotent. Put $H_1 = H$, $H_r = N_G(H_{r-1})$ the normaliser of H_{r-1} for $r > 1$. By nilpotence there is an $n \in \mathbb{N}$ such that H_n is finite and H_{n+1} is infinite. Since $N_G(H_n) / C_G(H_n)$ is finite, $C_G(H_n)$ is infinite. Hence $C_G(H_n)^{o} = G^{o}$ as G is one dimensional. Hence G^{o} centralises $H \leqslant H_n$. $G = HG^{o}$ and G^{o} is abelian so $G^{o} \leqslant Z(G)$.

<u>Step 5</u> WLOG $H\cap G^{o}$ may be identified with a finite subfield of K.

<u>Proof</u> We identify G^{o} with K^{+} and denote by $K(q)$ the set of elements of G^{o} which correspond to the subfield of K having q elements, where q is some power of p.

$H \cap G^o$ is a finite subgroup of G^o, thus $H \cap G^o \leqslant K(q)$ for some q. Put $\check{H} = HM$ where $M = K(q)$; $\check{H} \cap G^o = M$ moreover $V \mid \alpha_o(\alpha^o(V))$, where $\alpha : K[\check{H}] \to K[H]$ is the restriction map, since H is finite. Hence if $\alpha^o(V)$ is a finite G component, so is V.

<u>Step 6</u> V is a finite G component.

<u>Proof</u> V affords representation $\rho : H \to End_K(V)$ say. Since $H \cap G^o = K(q)$ we get by the above lemma

$$\rho|_{H \cap G^o} : H \cap G^o . \to End_K(V) \text{ extends to}$$

$$\rho_1 : G^o \to End_K(V).$$

There is an epimorphism of algebraic groups $\pi : H \times G^o \to G$, $\pi(x,y) = xy$ for $(x,y) \in H \times G$. $Ker\,\pi = \{(x,x^{-1}) \mid x \in H \cap G^o\}$. Define a representation $\sigma_1 : H \times G^o \to End_K(V)$ by $\sigma_1(x,y) = \rho(x)\rho_1(y)$ for $(x,y) \in H \times G^o$.

Since $G^o \leqslant Z(G)$ this defines a rational representation of $H \times G^o$. Now $\sigma_1(ker\,\pi) = 1$ and so σ_1 determines a rational representation $\sigma : G \to End_K(V)$ which extends V. Hence if σ affords the G module Y, $\varphi_o(Y) \cong V$ and so V is a finite G component.

This proposition clearly proves the claim made at the begining of the section.

<u>Remarks</u> 1 We wonder whether it is true that if H is a finite subgroup of an algebraic group G over K then every finite dimensional KH module is a finite G component.

2 By analogy with Green's theorem [5, theorem 8] and the theory of relative projectivity for the finite dimensional modules of a finite group we had hoped that a proof of the conjecture of 3.1 would shed light on the theory of relative injectivity for algebraic groups, in the sense of [8]. We no longer believe this would be the case.

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